ON THE EXTREME POINTS OF CONVEX SETS

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Introduction. A convex set is a set such that if it contains two points, it contains the segment joining these points [1, p. 2, and 2. Numbers in square brackets refer to the bibliography at the end]. Minkowski defined certain points of convex sets which he called extreme points [1, pp. 15–16; 3, p. 157]. They are related to certain other points which are here called *extreme points in the sense of distance* to distinguish them from the former, which are called *extreme points in the sense of Minkowski*. A detailed study is made of these two types of extreme points of convex sets in abstract normed linear spaces.

In the first place, it is necessary to distinguish two types of normed linear spaces on the basis of the convexity properties of spherical neighborhoods (§1). A normed linear space such that the segment joining any two points of a spherical neighborhood is interior to the neighborhood except at most for the given points themselves is called a space L^*. All other normed linear spaces are classed together and denoted by L. The study of extreme points is far simpler in spaces L^* than in spaces L, and the results are more complete. An example considered in §10 shows that the property of being a space L^* may depend on the properties of the distance function alone and not on the linearity properties of the space.

In §2 the existence of extreme points is considered. An approximation theorem first proved by Minkowski for euclidean 3-space is extended to spaces L^* and L in §3. Two kinds of convex sets are distinguished in §4 on the basis of the relation of the two kinds of extreme points, and it is shown that the set of extreme points in the sense of Minkowski may be either closed or not closed. In §5 the closed convex hull of a given set is considered, and Minkowski's Approximation Theorem (§3) is extended.

A general theorem on compact sets is established in §6. It is shown that in a complete metric space a set is compact if it is possible to approximate uniformly to it by means of closed compact sets. This theorem and Minkowski's Approximation Theorem (§5) enable us to show in §7 that the closed convex hull of a compact set in a Banach space is compact.

The significance of Minkowski's Approximation Theorem is considered briefly in §8. A series of theorems is given in §9 which establish more precisely the relation between a convex set and its extreme points. Some examples are considered in §10.

The paper may be considered a study in the geometry of abstract space.

1. **Linear spaces and extreme points.** A space which is linear and normed will be designated by L, its elements or points represented by x, y, $\cdots$, and

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