THE MAPS OF AN n-COMPLEX INTO AN n-SPHERE

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1. **Introduction.** The classes of maps of an n-complex into an n-sphere were classified by H. Hopf[1] in 1932. Recently, W. Hurewicz[2] has extended the theorem by replacing the n-sphere by much more general spaces. Freudenthal[3] and Steenrod[4] have noted that the theorem and proof are simplified by using real numbers reduced mod 1 in place of integers as coefficients in the chains considered. We shall give here a statement of the theorem which seems the most natural; the proof is quite simple. As in the original proof by Hopf, we shall base it on a more general extension theorem.

The fundamental tool of the paper is the relation of "coboundary";[5] it has come into prominence in the last few years.

In later papers we shall classify the maps of a 3-complex into a 2-sphere and of an n-complex into projective n-space.

I. Elementary facts

2. **Boundaries and coboundaries.** Let K be a complex, with oriented cells σ_i^r (not necessarily simplicial) of dimension r, $r = 0, \cdots, n$. Let $\partial_{ij}^r = 1, -1$, or 0 according as σ_i^{r-1} is positively, negatively, or not at all, on the boundary of σ_j^r. An r-chain C^r is a linear form $\Sigma\alpha_i\sigma_i^r$, the α_i being integers (or elements of an abelian group). The *boundary* (or *contraboundary*) and *coboundary* of C^r are defined by

$$\partial\Big(\sum_i \alpha_i\sigma_i^r\Big) = \sum_{i,j} \alpha_i\partial_{ji}^r\sigma_j^{r-1}, \qquad \delta\Big(\sum_i \alpha_i\sigma_i^r\Big) = \sum_{i,j} \alpha_i\partial_{ij}^{r+1}\sigma_j^{r+1}. \tag{2.1}$$

As in the ordinary theory, we say C^r is a *cocycle* if its coboundary vanishes, and C^r is *cohomologous* to D^r, $C^r \backsim D^r$, if $C^r - D^r$ is a coboundary. The relation $\delta\delta C^r = 0$ (easily proved; equivalent to $\partial\partial C^r = 0$) says that every coboundary

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[1] H. Hopf, Commentarii Mathematici Helvetici, vol. 5 (1932), pp. 39–54. See also Alexandroff-Hopf, *Topologie* I, Ch. XIII. A recent proof has been given by S. Lefschetz, Fund. Math., vol. 27 (1936), pp. 94–115. In Lemma 3 he gives a new proof of the theorem of the preceding paper; the author does not understand how the final map is made simplicial.

[2] W. Hurewicz, Proc. Kön. Akad. Wet. Amsterdam, vols. 38–39 (1935–36); in particular, vol. 39, pp. 117–126. The full paper will appear in the Annals of Math.

[3] H. Freudenthal, Compositio Math., vol. 2 (1935), footnote 8.

[4] Unpublished.

[5] This is discussed briefly in §2. For further details, see our paper *On matrices of integers*, pp. 35–45 of this volume of this Journal. We refer to this paper as I. The relation of Theorems 2, 3 and 4 to the theorems as stated by Hopf are made apparent by the theorems in I. *The present paper is independent of* I.