ON THE MAPS OF AN n-SPHERE INTO ANOTHER n-SPHERE

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1. **Introduction.** It is well known that to each map[1] f of an n-sphere S^n into another one S_0^n ($n \geqq 1$ always) there corresponds a number d_f, the *degree* of f, and $d_f = d_g$ if f and g are homotopic (see §2). H. Hopf[2] has proved the converse theorem, that if $d_f = d_g$, then f and g are homotopic. The object of this note is to give an elementary proof of the latter theorem. The methods will be used and extended in later papers.

In an appendix we give somewhat briefly a proof of the theorem for the case that $d_f = 0$. This is the only case needed in the following paper; the general theorem then follows from that paper. The second proof is more intuitive geometrically than the first, but complete details would make it perhaps more lengthy.

2. **On deformations.** A *deformation* of one space S in another S_0 is a family $\phi_t(p)$ ($0 \leqq t \leqq 1$, p in S) of maps of S into S_0, continuous in both variables together. Given maps f and g of S into S_0, if there exists a deformation ϕ_t such that $\phi_0 \equiv f$ and $\phi_1 \equiv g$, we say f and g are *homotopic*. If f is homotopic to g, where $g(p) \equiv P_0$ (all p in S), we say f is *homotopic to zero*, and f may be *shrunk to the point* P_0.

Suppose S and S_0 are complexes, K_0 is a simplicial subdivision of S_0, and f maps S into S_0. Then, for a sufficiently fine simplicial subdivision K of S, the following is true. To each vertex V of K we may choose a vertex $g(V)$ of a cell of K_0 which contains $f(V)$, so that the vertices of any cell of K go into the vertices of a cell of K_0. This determines uniquely a "simplicial map" g of K into K_0, affine in each cell (see §5); moreover, f is homotopic to g.

3. **The degree of a map.** Let S_0^n be the unit n-sphere in $(n + 1)$-space, let K_0^n be a simplicial triangulation of S_0^n, and let σ_0^n be an n-cell of K_0^n. We choose K_0^n so that if P_1 is a point of σ_0^n and P_0 is the antipodal point of S_0^n, each great semicircle from P_1 to P_0 intersects the boundary $\partial\sigma_0^n$ of σ_0^n in exactly one point. By pushing along these semicircles, we define a deformation Ω_t of the identity $\Omega_0(p) \equiv p$ into a map Ω_1, where $\Omega_1(p) \equiv P_0$ for p in $S_0^n - \sigma_0^n$.

Let σ^k be a k-cell ($k \leqq n$), in fixed correspondence with a k-simplex, and let

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[1] All maps will be assumed continuous.

[2] See Alexandroff-Hopf, *Topologie*, I, Berlin, 1935, pp. 501–505. See also the reference to Lefschetz in the following paper.