FUNCTIONS REPRESENTABLE BY TWO LAPLACE INTEGRALS

By D. H. Ballou

1. Introduction. One of the properties of the Laplace integral representation of a function $f(z)$,

$$f(z) = \int_0^\infty e^{-zt}\varphi(t)\,dt, \tag{1}$$

is the uniqueness of the determining function $\varphi(t)$ when that function is continuous.[1] It has been pointed out by G. Doetsch[2] that if a function $f(z)$ may be expanded into two different series the terms of which are representable by Laplace integrals and if the term by term transformation of those series is permissible, then this function is represented by two different Laplace integrals. Furthermore, there will follow from the uniqueness property the equality of two new series, the determining functions of the integrands.

It has been known that the cotangent was one function capable of such a representation,[3] for it has series developments both in terms of partial fractions and of exponentials:

$$\operatorname{ctn} z = \frac{1}{z} + 2\sum_{n=1}^{\infty} \frac{z}{z^2 - n^2\pi^2}, \tag{2}$$

$$\operatorname{ctn} z = -i\left(1 + 2\sum_{n=1}^{\infty} e^{2niz}\right). \tag{3}$$

Now if we take the function[4] $\dfrac{\operatorname{ctn}\sqrt{-s}}{-\sqrt{-s}}$ the terms of the series are representable

Received May 11, 1936; presented to the American Mathematical Society, September 1, 1936.

[1] See, for instance, D. V. Widder, *A generalization of Dirichlet's series and Laplace's integrals by means of a Stieltjes integral*, Transactions of the American Mathematical Society, vol. 31 (1929), p. 705.

[2] G. Doetsch, *Überblick über Gegenstand und Methode der Funktionanalysis*, Jahresbericht der Deutschen Mathematiker-Vereinigung, vol. 36 (1927), p. 28.

[3] G. Doetsch, loc. cit. See also H. Hamburger, *Über einige Beziehungen, die mit der Funktionalgleichung der Riemannschen ζ-Funktion äquivalent sind*, Mathematische Annalen, vol. 85 (1922), p. 129.

[4] Here and throughout this paper that branch of the double-valued function $z = \sqrt{-s}$ is taken which corresponds to the upper half of the z-plane. We then are dealing only with single-valued functions and we are assured of the convergence of (5) for $R(s) > 0$.