INEQUALITIES AMONG THE INVARIANTS OF PFAFFIAN SYSTEMS

BY DONALD C. DEARBORN

1. Introduction. Associated with any pfaffian system

$$S\colon \omega^\alpha = a_i^\alpha \, dx^i = 0 \qquad (\alpha = 1, 2, \cdots, r; i = 1, 2, \cdots, n)$$

are certain arithmetic invariants. Among these are the number r of independent equations in the system, the species σ, the class p, and the half-rank ρ.[1] These invariants are all non-negative integers.

The object of this paper is to find sets of inequalities which must be satisfied by these four invariants for any pfaffian system. If for every non-negative integral solution of such a set of inequalities it is possible to find a pfaffian system having that solution as its invariants, the set of inequalities will be called *complete.*

In §2 sets of inequalities are found which hold for any pfaffian system. These sets are not, in general, complete sets. In §3 a complete set of inequalites is given for systems having equal species and half-rank. Included in this classification are all completely separable[2] systems, such as passive systems, systems consisting of a single equation, and systems having $r - 1$ integrals. Systems having rank two are considered in §4. It is shown that such systems have species one or two, and complete sets of inequalities are obtained.

2. Inequalities satisfied by the invariants of any system. It is known that[3] $\rho \leqq \sigma$ and that[4] $p \geqq r + \sigma + 1$ unless the system is passive.

Since there are r independent equations in S, the system may be solved algebraically for r of the differentials and put in the reduced form[5]

$$(2.1) \qquad \omega^\alpha = dx^\alpha + A_\lambda^\alpha \, dx^\lambda \qquad (\alpha = 1, \cdots, r; \lambda = r + 1, \cdots, r + \sigma),$$

where S is assumed to be expressed in terms of the minimum number of differentials. The derived forms are then $\omega'^\alpha = dA_\lambda^\alpha \, dx^\lambda$, which we write as

Received June 4, 1936; part of a doctoral dissertation presented June, 1936 at Duke University.

[1] For definition of class see E. Goursat, *Leçons sur le Problème de Pfaff*, Paris, 1922, p. 268. For species see J. M. Thomas, *Pfaffian systems of species one*, Trans. Amer. Math. Soc., vol. 35 (1933), pp. 356–371. For half-rank see E. Cartan, *Invariants Intégraux*, Paris, 1922, p. 59; Mabel Griffin, *Invariants of pfaffian systems*, Trans. Amer. Math. Soc., vol. 35 (1933), p. 931.

[2] Griffin, loc. cit., p. 936.

[3] J. M. Thomas, *A lower limit for the species of a pfaffian system*, Proc. Nat. Acad. Sci., vol. 19 (1933), p. 913.

[4] Thomas, loc. cit., footnote 1.

[5] Thomas, loc. cit., footnote 1, p. 362.