TRIPLES OF CONJUGATE HARMONIC FUNCTIONS AND MINIMAL SURFACES

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A surface S is said to be given in terms of isothermic parameters u, v, provided the representation

$$S\colon x_j = x_j(u, v), \qquad j = 1, 2, 3, \qquad (u, v) \text{ in } D, \tag{1}$$

where D is some finite domain of definition, is such that

$$E = G = \lambda(u, v), \qquad F = 0, \tag{2}$$

where

$$E = x_{1,u}^2 + x_{2,u}^2 + x_{3,u}^2, \qquad F = x_{1,u}x_{1,v} + x_{2,u}x_{2,v} + x_{3,u}x_{3,v},$$
$$G = x_{1,v}^2 + x_{2,v}^2 + x_{3,v}^2,$$

the second subscripts denoting differentiation. Such a representation is conformal except where $\lambda(u, v) = 0$.

A theorem of Weierstrass states that a necessary and sufficient condition that a surface S, given in terms of isothermic parameters, be minimal is that the coördinate functions be harmonic. Then in any simply connected part of D, the functions x_j are the real parts of analytic functions,

$$x_j = \Re f_j(w), \qquad w = u + iv,$$

and (2) is equivalent to

$$\sum_{j=1}^{3} f_j'^2(w) = 0. \tag{3}$$

If an isothermic representation (1) of the minimal surface S is such that one of the coördinate functions is identically zero, say $x_3(u, v) \equiv 0$, then either $x_1(u, v) + ix_2(u, v)$ or $x_2(u, v) + ix_1(u, v)$ is an analytic function of the complex variable $w = u + iv$, and $x_1(u, v)$ and $x_2(u, v)$ are said to form a *couple of conjugate harmonic functions*. By analogy, the coördinate functions of any minimal surface in isothermic representation have been called a *triple of conjugate harmonic functions*.[1]

The analogy here indicated between analytic functions of a complex variable and isothermic representations of minimal surfaces has often been noted, and since the time of Weierstrass has served as a guiding principle in the study of minimal surfaces. It is the purpose of the present paper to pursue this analogy from the coefficients viewpoint.

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[1] E. F. Beckenbach and T. Radó, *Subharmonic functions and minimal surfaces*, Trans. Amer. Math. Soc., vol. 35 (1933), pp. 648–661.