SEMI-CLOSED SETS AND COLLECTIONS

By G. T. Whyburn

1. A set K in a metric space S will be said to be *semi-closed* provided each component of K is closed and any convergent sequence of components of K whose limit set intersects $S - K$ converges to a single point of $S - K$.

Similarly, a collection G of disjoint sets is said to be semi-closed if each set of G is closed and any convergent sequence of sets of G whose limit set intersects $S - G^*$ converges to a single point of $S - G^*$, where G^* denotes the point set which is the sum of all the sets of the collection G.

For example, any closed set is semi-closed, as is also any totally disconnected set or the sum of any closed set and any set of dimension zero. Any null collection of disjoint closed sets (i.e., a collection having only a finite number of elements of diameter greater than any preassigned $\epsilon > 0$) is semi-closed. The collection of components of any closed set K is semi-closed, as is also this collection together with an arbitrary null collection of disjoint closed sets, no one of which intersects K.

The principal object of the present paper will be to develop conditions under which the complements of semi-closed sets and collections in various continuum spaces will be connected and locally connected.

2. We begin with some results giving fundamental relations between these sets and collections and upper semi-continuous collections.[1]

(2.1) Theorem. *If a collection G of disjoint closed sets is upper semi-continuous, then in order that G be semi-closed it is necessary and sufficient that the decomposition of S into the sets of G and the individual points of $S - G^*$ be upper semi-continuous.*

This theorem follows immediately from the definitions of semi-closed and of upper semi-continuous collections.

(2.11) Corollary. *If G is any upper semi-continuous collection of disjoint closed sets filling up S, and if G_0 is the set of all non-degenerate elements of the collection G, then any subcollection G_1 of G such that $G_0 \subset G_1 \subset G$ is semi-closed.*

(2.2) *The collection G of all components of any semi-closed set K in a compact space S is upper semi-continuous.*

For if this were not so, there would exist a convergent sequence $g_1, g_2, \cdots$ of sets of G such that if $L = \lim (g_i)$, then for some $g \epsilon G$ we have

$$L \cdot g \neq 0 \neq L \cdot (S - g).$$

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[1] See R. L. Moore, Transactions of the American Mathematical Society, vol. 27 (1925), pp. 416–428. As used in the present paper, a collection G is upper semi-continuous provided that for every convergent sequence of elements (g_i) of G whose limit set L intersects $g \epsilon G$ we have $L \subset g$. For compact spaces this is equivalent to Moore's original definition. See ref. 3.