SYMMETRIC FUNCTIONS OF NON-COMMUTATIVE ELEMENTS

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Introduction. A study of symmetric polynomials of matrices, for which the commutative law of multiplication need not necessarily be valid, led to the study of symmetric polynomials of certain abstract elements for which the processes of addition and multiplication obey the postulates of a linear associative algebra. This results in a generalization of the definition of the elementary symmetric functions. For example, if x_1 and x_2 are such elements, let x_1x_2 symbolize x_1 multiplied on the right by x_2 and let $x_1 + x_2$ indicate addition of x_1 and x_2; then since x_1x_2 differs in general from x_2x_1, the second elementary symmetric function of the elements x_1 and x_2 becomes

$$E_2 = \Sigma x_1x_2 = x_1x_2 + x_2x_1;$$

but as before, $E_1 = \Sigma x_1 = x_1 + x_2$. The simple symmetric functions of third degree of the elements x_1 and x_2 are

$$\Sigma x_1x_2x_1 = x_1x_2x_1 + x_2x_1x_2, \qquad \Sigma x_1x_2^2 = x_1x_2^2 + x_2x_1^2,$$
$$\Sigma x_1^2x_2 = x_1^2x_2 + x_2^2x_1, \qquad \Sigma x_1^3 = x_1^3 + x_2^3.$$

These functions cannot be expressed as polynomials in E_1 and E_2 as in the case of commutative elements, but another polynomial, for example $\Sigma x_1x_2x_1$, must be defined as an elementary symmetric function in addition to E_1 and E_2 if the fundamental theorem is to be reëstablished. Note also that E_1E_2 differs from E_2E_1 for non-commutative elements. If three elements x_1, x_2, x_3 are considered, two polynomials of third degree are required to serve as elementary symmetric functions instead of the one function $E_3 = \Sigma x_1x_2x_3$ of the commutative elements. Two polynomials which may be used are $E_3 = \Sigma x_1x_2x_3$ and $\Sigma x_1x_2x_1$. This paper shows that as the number of elements and the degree are increased, an infinite sequence of symmetric polynomials, consisting of a finite set of one or more for each degree can be chosen so that every symmetric polynomial may be expressed uniquely in terms of the polynomials of this sequence and the coefficients of the original polynomial, with coefficients which are integral. This sequence may be chosen in more than one way but the number for each degree is unique.

Since by the Poincaré equivalence theorem every linear associative algebra is equivalent to a matric algebra, no generality is lost if the elements are taken as matrices.

1. Simple symmetric polynomials and elements completely non-commutative of order m. The usual definitions and theorems which apply to sym-

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