NOTE ON A SINGULAR INTEGRAL. II

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1. **Introduction.** This paper is concerned with the convergence in the mean to $f(x)$, as $m \to \infty$, of the integral

$$T_m(x;f) = (2\pi)^{-\frac{1}{2}} \int_{-\infty}^{+\infty} K(x-u;m) f(u)du,$$

and is a generalization of results obtained in an earlier note by the author.[1] As a point of departure for the present note we shall, after a few preliminary remarks, introduce the main theorem (hereafter referred to as Theorem I) of the first one.

Since all of the functions to be considered will be defined over the infinite range, we shall denote the Lebesgue class $L_r(-\infty, +\infty)$ by simply L_r. We write $\| f(x) \|_r$ for the norm of a function in L_r, and define it by means of the relation

$$\| f(x) \|_r \equiv \left[\int_{-\infty}^{+\infty} | f(x) |^r dx \right]^{\frac{1}{r}}.$$

The Fourier transform of a function $f(x) \epsilon L_r$, $r > 1$, is defined (provided it exists) as the limit in the mean of order s, $1/r + 1/s = 1$, as $A \to \infty$, of the integral

$$(2\pi)^{-\frac{1}{2}} \int_{-A}^{A} e^{-ixt} f(t)dt,$$

and will be denoted by $T[x; f(t)]$, or, if there can be no confusion regarding the argument, more simply by $T[f(x)]$. The inverse Fourier transform of $f(x)$, denoted by $T^{-1}[x; f(t)] \equiv T^{-1}[f(x)]$, is defined by the same expression, except that e^{-ixt} is replaced by its complex conjugate.

THEOREM I. *Let $K(x; m) \epsilon L_2$ for every m. Then in order that $T_m(x; f) \epsilon L_2$ for every m and $\| T_m(x; f) - f(x) \|_2 \to 0$ as $m \to \infty$, for every $f(x) \epsilon L_2$, it is necessary and sufficient that $K(x; m)$ satisfy the conditions*

$$\text{(i)} \qquad \underset{-\infty < x < +\infty}{\text{e.l.u.b.}} | T[K(x;m)] | = M_m, \quad \textit{and} \quad \overline{\lim_{m\to\infty}} M_m < M,$$

$$\text{(ii)} \qquad \lim_{m\to\infty} \int_a^b | T[K(x;m)] - 1 |^2 dx = 0$$

for every finite a and b.

Remarks. In condition (i), as throughout the rest of the paper, M_m is a

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[1] Bull. Amer. Math. Soc., vol. 40 (1934), pp. 494–496.