REPRESENTATION OF POSITIVE HARMONIC FUNCTIONS

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We are concerned with the problem of representing the positive harmonic functions in a given region, and are primarily interested here in pointing out the relevance to this problem of a number of other problems, some of which have been discussed in the literature. The representation, by means of the Poisson-Stieltjes integral, of the positive harmonic functions in a sphere is an instance of the type of representation with which we are concerned. The analytical technique customarily employed in establishing the Poisson-Stieltjes representation or one of its generalizations requires relatively stringent smoothness conditions (e.g., bounded curvature) upon the boundary of the region.[1] It is true that the criteria we here cite as sufficient for a solution of the representation problem are less explicitly connected with the nature of the boundary than are the usual conditions just referred to, and it does not seem a trivial problem to characterize intrinsically the regions for which these criteria are satisfied. Nevertheless, as we shall show elsewhere, our criteria are satisfied by classes of regions considerably broader than those to which the customary technique applies. This would seem to make it clear that the representation problem does not depend essentially on smoothness conditions, even in three or more dimensions where conformal mapping no longer serves as a *deus ex machina*.

In the present note we shall point out the criteria and give one two-dimensional application: a direct representation—that is, a representation not depending upon the intervention of conformal mapping—of the positive harmonic functions in a finitely multiply connected Jordan region. For simplicity we shall restrict the discussion to bounded regions and shall use two-dimensional language, but it is to be emphasized that, except in the application at the end, the argument is independent of the number of dimensions.

The representation in question is of the form

$$u(P) = \int_{A^*} f(S, P)\, d\mu(e_S), \tag{1}$$

where $u(P)$ is a non-negative harmonic function in a bounded region A, where A^* is the frontier of A, where $f(S, P)$ is a certain function which depends only upon the region A and which is defined for $S \epsilon A^*$, $P \epsilon A$, and where $\mu(e)$ is a finite, non-negative, completely additive function of Borel sets which vanishes

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[1] de la Vallée Poussin, *Propriétés des fonctions harmoniques dans un domaine ouvert limité par des surfaces à courbure bornée*, Annali della R. Scuola Normale Superiore di Pisa, (2), vol. 2 (1933), pp. 167–197; George A. Garrett, *Necessary and sufficient conditions for potentials of single and double layers*, Am. Jour. of Math., vol. 58 (1936), pp. 95–129.