ON THE CLOSURE OF $\{e^{i\lambda_n x}\}$

By Norman Levinson

1. A set $\{e^{i\lambda_n x}\}$ is said to be closed $L^p(-\pi, \pi)$ if for any $f(x) \in L^p(-\pi, \pi)$

$$\int_{-\pi}^{\pi} f(x)e^{i\lambda_n x}\, dx = 0 \tag{1.0}$$

implies that $f(x)$ is equivalent to zero.

Here we will concern ourselves with the closure properties of the set

$$\{e^{i\lambda_n x}\} \qquad (-\infty < n < \infty),$$

where

$$\lim_{|n| \to \infty} \frac{n}{\lambda_n} = 1, \tag{1.1}$$

that is, the λ_n are positive for sufficiently large $n > 0$ and have density 1,[1] and a corresponding result holds for $n < 0$.

The question of the closure of such sets was first investigated by Wiener and Paley,[2] who considered closure in $L^2(-\pi, \pi)$ of even sets ($\lambda_{-n} = -\lambda_n$). They made a special study of the set $\{1, e^{\pm i\lambda_n x}\}$, $n > 0$, where

$$|\lambda_n - n| \leqq B, \qquad n > 0, \tag{1.2}$$

and showed that if $B < \frac{1}{8}$, the set is closed $L^2(-\pi, \pi)$. Here it will be shown that for closure $L^2(-\pi, \pi)$ it suffices that $B \leqq \frac{1}{4}$.

First we shall obtain a general closure criterion. We shall use this criterion to get results under conditions of the type (1.2) and we shall show that these results are the best possible.

Our basic criterion is given by

Theorem I. *Let $\{\lambda_n\}$ satisfy (1.1). Let $\Lambda(u)$ be the number of $|\lambda_n| \leqq u$. If*

$$\int_1^v \frac{\Lambda(u)}{u}\, du > 2v - \frac{p-1}{p} \log v - C \tag{1.3}$$

for some constant C, the set $\{e^{i\lambda_n x}\}$, $-\infty < n < \infty$, is closed $L^p(-\pi, \pi)$, $p \geqq 1$.

A corollary of Theorem I is

Theorem II. *If*

$$|\lambda_n - n| \leqq \tfrac{1}{2} N + \frac{p-1}{2p} \qquad (-\infty < n < \infty), \tag{1.4}$$

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[1] If the density is different from 1 (and not zero or infinite), the problem is reducible to this one by making a change of scale. In case densities do not exist, see Levinson, Proc. Camb. Phil. Soc., vol. 31 (1935), pp. 335–346.

[2] Wiener and Paley, *Fourier Transforms in the Complex Domain*, Am. Math. Soc. Coll. Pub., vol. XIX, Chap. VI.