A CONSTRUCTION FOR PRIME IDEALS AS ABSOLUTE VALUES OF AN ALGEBRAIC FIELD

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1. **Introduction.** The difficulties of actually constructing the prime ideal factors of a rational prime p in an algebraic field have had a considerable influence upon the development of ideal theory. One of the most practical of the methods for this construction consists of three successive "approximations" to the prime factors of p in terms of certain Newton Polygons, similar to the polygons used in the expansion of algebraic functions. This method, due to Ore,[1] is directly applicable in all but certain exceptional cases. The present paper extends the method to all cases by making not three but any number of successive approximations. To formulate this extension simply, it is necessary to replace the prime ideals by certain corresponding "absolute values", which succinctly express the essential properties of the Newton polygons. In terms of these values, the successive approximations are a natural application of a method of finding possible "absolute values" for polynomials.

To introduce these absolute values, consider the ring $\mathfrak{o}$ of all algebraic integers of an algebraic number field, and let $\mathfrak{p}$ be a prime ideal in $\mathfrak{o}$. Since every integer α of the field can be written in the form $(\alpha) = \mathfrak{p}^m \cdot \mathfrak{b}$, where $\mathfrak{b}$ is an ideal prime to $\mathfrak{p}$, we can write the exact exponent m to which $\mathfrak{p}$ divides α as a function $W\alpha = m$. Because of the unique decomposition theorem,

$$(1) \qquad W(\alpha \cdot \beta) = W\alpha + W\beta, \qquad W(\alpha + \beta) \geqq \min (W\alpha, W\beta).$$

Any function $V\alpha$ which has these two properties is called a non-archimedean value or a "Bewertung"[2] of the ring $\mathfrak{o}$, while the particular function W obtained from $\mathfrak{p}$ may be called a $\mathfrak{p}$-adic value. Every value V of $\mathfrak{o}$ is a constant multiple[3] of some $\mathfrak{p}$-adic value W. Hence absolute values can replace prime ideals.

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[1] O. Ore, *Zur Theorie der algebraischen Körper*, Acta Math., vol. 44 (1924), pp. 219–314; O. Ore, *Newtonsche Polygone in der Theorie der algebraischen Körper*, Math. Annalen, vol. 99 (1928), pp. 84–117. These papers will be cited as Ore I and Ore II, respectively.

[2] W. Krull, *Idealtheorie*, Ergebnisse der Mathematik und ihrer Grenzgebiete, Bd. 4, Heft 3. This text, cited henceforth as Krull I, contains further references on absolute values.

[3] E. Artin, *Ueber die Bewertungen algebraischer Zahlkörper*, Jour. für Math., vol. 167 (1932), pp. 157–159. The theorem may be proved thus: Given V, first show that any rational integer $n = 1 + 1 + \cdots + 1$ has a non-negative value and then from (1) that every algebraic integer has a non-negative value. If the value of an ideal $\mathfrak{b}$ be defined as the minimum of $V\alpha$ for $\alpha \in \mathfrak{b}$, then one and only one prime ideal $\mathfrak{p}$ can have a positive value, and V must be $\mathfrak{p}$-adic. A similar theorem holds when $\mathfrak{o}$ is an abstract ring in which the usual prime-ideal decomposition holds. (B. L. van der Waerden, *Moderne Algebra*, vol. 2, §100.)