THE ALMOST PERIODIC BEHAVIOR OF THE FUNCTION $1/\zeta(1 + it)$

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It is known[1] that the prime-number theorem implies the convergence of the development

$$(1) \qquad 1/\zeta(s) = \sum_1^\infty \mu(n)\, n^{-s},$$

which is obvious in the half-plane $\sigma > 1$, at every point of the line $\sigma = 1$ also. The object of the present note is to show that the *trigonometrical* series

$$(2) \qquad 1/\zeta(1 + it) = \sum_1^\infty \mu(n) n^{-(1+it)} = \sum_1^\infty \mu(n) n^{-1} \exp(-it \log n)$$

is the *Fourier* series of the function which it represents, i.e., that

$$(3) \qquad 1/\zeta(1 + it) \sim \sum_1^\infty \mu(n) n^{-1} \exp(-it \log n),$$

where the sign $\sim$ refers to the class B^2 of Besicovitch.[2] In other words, the function $1/\zeta(1 + it)$ is almost periodic (B^2), and, on placing

$$\mathfrak{M}\{f(t)\} = \lim_{T=+\infty} \mathfrak{M}_T\{f(t)\},$$

where

$$(4) \qquad \mathfrak{M}_T\{f(t)\} = \int_0^T f(t) dt/T,$$

the mean value

$$(5) \qquad \mathfrak{M}\{e^{i\lambda t}/\zeta(1 + it)\}$$

exists for every real λ and is 0 or $\mu(n)/n$ according as $\lambda \neq \log n$ or $\lambda = \log n$, where $n = 1, 2, \cdots$. On choosing $n = 1$, it follows, in particular, that

$$(6) \qquad \mathfrak{M}\{1/\zeta(1 + it)\}$$

exists and is equal to $\mu(1) = 1$.

Since (3) refers to the class (B^2), it also follows that $\mathfrak{M}\{|\zeta(1 + it)|^{-2}\}$ exists. The latter result, proved by Landau on pp. 801–804 of his *Handbuch*, suggests but does not imply (3); it does not even imply the existence of the Fourier constants (5), (6).

Received December 28, 1935.

[1] Cf. p. 811 of the article by Bohr and Cramér in vol. 2, III_2 of the *Encyklopädie der mathematischen Wissenschaften*, where several references are given.

[2] A. S. Besicovitch, *Almost Periodic Functions*, Cambridge, 1932, Chap. II.