NEW THEOREMS AND METHODS IN DETERMINANT THEORY

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Introduction. If to each ordered pair of undefined elements p, q of an abstract space (set) S, a real, non-negative number pq can be attached such that $pq = qp$, and $pq = 0$ if and only if p is identical with q, the space S is said to be *semimetric*. The elements p, q may be spoken of as points of the space, with pq as their distance. A given semimetric space S is *characterized metrically* when conditions are stated (in terms of distance relations) which are necessary and sufficient for any semimetric space satisfying them to be mapped isometrically (congruently) upon S. Among those semimetric spaces which have been characterized metrically are the n-dimensional euclidean,[1] spherical,[2] and hyperbolic[3] spaces.

In this paper results obtained in the metric characterization of these spaces are introduced for the purpose of deriving new theorems concerning certain types of symmetric determinants. The application of isometric geometry to determinant theory furnishes a new and powerful impetus for its development. By such an application one obtains elegant proofs of novel and interesting theorems. These new methods are well adapted (1) for proving whole chains of theorems, as in §§1 and 5, (2) for the determination of relations between the elements of a determinant, as in Theorems 3.1 and 5.2, and (3) for ascertaining the sign of certain determinants, whose elements are not explicitly known.

While only determinants with real elements are treated in this paper, the development of the theory of complex metric spaces, already under way, may be expected to furnish results that can be applied to determinants with complex elements.[4]

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[1] Menger, *Untersuchungen über allgemeine Metrik*, Mathematische Annalen, vol. 100 (1928), pp. 75–163. This paper is divided into three parts; the second part (Zweite Untersuchung, pp. 113–141) contains a characterization of the n-dimensional euclidean space in terms of relations between the distances of its points. In a later paper, *New foundation of euclidean geometry*, American Journal of Mathematics, vol. 53 (1931), pp. 721–745, the concept of quasi-congruence order is introduced.

[2] Blumenthal, *Concerning spherical spaces*, American Journal of Mathematics, vol. 57 (1935), pp. 51–61. See also vol. 55 (1933), pp. 619–640, as well as L. Klanfer, *Metrische Charakterisierung der Kugel*, Ergebnisse eines mathematischen Kolloquiums, Wien, Heft 4 (1933), pp. 43–45.

[3] Blumenthal, *The metric characterization of the n-dimensional hyperbolic space*, Bull. Amer. Math. Soc., vol. 41 (1935), p. 485 (Abstract).

[4] A. Wald, *Komplexe und indefinite Räume*, Ergebnisse eines mathematischen Kolloquiums, Wien, Heft 5 (1933), pp. 32–42.