λ-DEFINABILITY AND RECURSIVENESS

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1. **Introduction.** In Kleene [2][1] a theory of the definition of functions of positive integers by certain formal means is developed in connection with the study of a system of formal logic.[2] The system of formal logic is shown in Kleene-Rosser [1] to be inconsistent; however, the theory of formal definition remains of interest, both for its use in a new system of formal logic proposed by Church in [3], and for its connection with questions of constructibility and decidability in number theory.[3] Hence it seems desirable to bring together the essentials of the theory, and to develop them from a somewhat new point of view, in which the emphasis is on the connection with the recursive functions. In this presentation, no knowledge of systems of formal logic is presupposed, but use will be made of a few results of the intuitive theory of recursive functions.[4]

It is found convenient here to treat the functions as functions of natural numbers, rather than of positive integers. This change can be regarded as a change merely in the notation.

The theory deals with a class of formulas composed of the symbols {, }, (,), λ, [,] and other symbols f, x, ρ, $\cdots$ called variables or *proper symbols*, where f, x, ρ, $\cdots$ is a given infinite list.

A formula is called *properly-formed* if it is obtainable from proper symbols by zero or more successive operations of combining **M** and **N** to form {**M**} (**N**) or λ**x**[**M**], where **x** is any proper symbol. An occurrence of a proper symbol **x** in a formula **F** is called *bound* or *free* according as it is or is not an occurrence in a properly-formed part of the form λ**x**[**M**]. By a free (bound) symbol of **F** is meant a proper symbol which occurs in **F** as a free (bound) symbol. A formula shall be *well-formed*, if it is properly-formed, and if, for each properly-formed part of the form λ**x**[**M**], where **x** is a proper symbol, **x** is a free symbol of **M**.

Heavy-typed letters will henceforth represent undetermined well-formed formulas under the convention that each set of symbols standing apart in which a heavy-typed letter occurs shall stand for a well-formed formula.[5] As abbre-

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[1] The numbers in brackets refer to the bibliography at the end.

[2] Use is made, directly or indirectly, of Church [1]–[2], Kleene [1], Rosser [1], Curry [1]–[3], Schönfinkel [1].

[3] See Kleene [2] p. 232, Church [4], and Church-Kleene [1].

[4] In writing this paper, I have profited from discussion of the subject with Dr. J. B. Rosser, and I also thank him for assistance with the manuscript.

[5] A detailed analysis of the structure of well-formed formulas, and of the implications of this convention, is given in Kleene [1] §§2, 3. The term "proper symbol" was introduced in place of "variable" in order to save the latter for use in another meaning in connection with the formal logics under consideration.