ON THE POISSON SUMMABILITY OF FOURIER SERIES

By Norman Levinson

1. Let $f(x)$ be a Lebesgue integrable function of period 2π, and let

$$\phi(x) = f(y + x) + f(y - x) - 2s.$$

It is well known that if

$$\lim_{\epsilon \to 0} \frac{1}{\epsilon} \int_0^{\epsilon} \left(1 - \frac{x}{\epsilon}\right)^{m-1} \phi(x)\, dx = 0, \tag{1.0}$$

then

$$\lim_{\epsilon \to 0} \frac{1}{\epsilon} \int_0^1 \phi(x)\, dx \int_0^1 (1 - z)^n \cos \frac{xz}{\epsilon}\, dz = 0 \tag{1.1}$$

for $n > m$, where (1.1) is the n-th Riesz mean of the Fourier series for $f(x)$ at $x = y$.

In his conversation class, Hardy carried this relation over to Poisson summability of Fourier series by proving in a very simple manner that

$$\lim_{\epsilon \to 0} \epsilon \int_0^1 \frac{\phi(x)}{x^2} e^{-\frac{\epsilon^2}{x^2}}\, dx = 0$$

implies the Poisson summability of the Fourier series of $f(x)$ at the point $x = y$, and conjectured that

$$\lim_{\epsilon \to 0} \epsilon \int_0^1 \phi(x)\, e^{-\left(\frac{\epsilon}{x}\right)^{1+b}} \frac{dx}{x^2} = 0$$

also implies the P summability for $b > 0$. We shall show this to be the case. We shall also show that there is another exponential kernel $\exp\,[-\,(x/\epsilon)^{1+b}]$, similarly related to P summability.

Our theorems are

Theorem 1. *Let $E(m, \alpha)$ represent*

$$\lim_{\epsilon \to 0} \frac{1}{\epsilon} \int_0^1 \left(\frac{x}{\epsilon}\right)^{\alpha} e^{-\left(\frac{x}{\epsilon}\right)^{1+m}} \phi(x)\, dx = 0, \qquad m > -1, \qquad \alpha \geqq 0, \tag{1.2}$$

and $P(m)$ represent

$$\lim_{\epsilon \to 0} \frac{1}{\epsilon} \int_0^1 \frac{\phi(x)}{\left(\frac{x}{\epsilon}\right)^{2(1+m)} + 1}\, dx = 0, \qquad m > -\frac{1}{2}, \tag{1.3}$$

where $\phi(x)$ is defined as above. Then $E(n, \alpha)$ for $n > m$ and $\alpha \geqq 0$, or $E(m, \alpha)$ for $\alpha > 0$ implies $P(m)$, while $P(m)$ implies $E(n, \alpha)$ for $m > n$.

Received October 7, 1935.