ON LOCAL BETTI NUMBERS

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1. Introduction. Several types of local Betti numbers have been introduced recently by Alexandroff[1] and by Čech.[2] The local invariants introduced in this paper were discovered during an attempt to define edge and kernel points of a compact metric space. Incidentally, they give a direct generalization of the notion of the order, at a point, of a 1-dimensional set.[3]

Section 2 consists of a list of theorems, a knowledge of which is necessary in the later sections. In §3 the numbers $\beta^i(a, M)$, $i \geqq 0$, are defined for each point a of a compact metric space M, and this definition is illustrated in §4 by examples. In §5 are given several definitions of edge and kernel points which lead to simple necessary conditions that a compact metric space be imbeddable in the compact euclidean space of the same dimension. §6 is devoted to the determination of the Borel class of the set of all points of M for which the numbers $\beta^i(a, M)$ satisfy certain inequalities.

In §7 the numbers $\beta^i(a, M)$ are related to the local connectedness of the set M, and also to that of its complement when M is considered as a subset of a euclidean space. In order to extend these theorems, certain auxiliary theorems on the addition of irreducible membranes are required, and these are given in §8. Their immediate consequences are then developed in §9.

There exist in the literature numerous characterizations of the plane, the closed 2-cell and 2-manifold. The majority of these are purely set-theoretic, excepting certain definitions of Whitney and van Kampen[4] which make use of mixed methods. We give below, in §10, a characterization of the 2-manifold in terms of the numbers $\beta^i(a, M)$. In §11 it is shown that a similar characterization can be given for the closed 2-cell and, in fact, for any 2-dimensional set obtained from a 2-manifold by the omission of a finite number of open 2-cells. In §12 necessary and sufficient conditions are given that every point of a locally compact metric space have a neighborhood homeomorphic with a 2-cell, and these are applied to give characterizations of the open 2-cell (or euclidean plane) and of the class of cylinder-trees.[5] The characterizations mentioned in this paragraph are of a purely combinatorial nature.

Received June 17, 1935; presented to the American Mathematical Society, April 19, 1935.

[1] *On local properties of closed sets*, Annals of Mathematics, vol. 36 (1935), pp. 1–35.

[2] *Sur les nombres de Betti locaux*, Annals of Mathematics, vol. 35 (1934), pp. 678–701.

[3] Menger, *Kurventheorie*, p. 96.

[4] van Kampen, *On some characterizations of 2-dimensional manifolds*, this journal, vol. 1 (1935), p. 87.

[5] Zippin, *On continuous curves and the Jordan curve theorem*, American Journal of Mathematics, vol. 52 (1930), pp. 331–350.