ON THE NEIGHBORHOOD OF A GEODESIC IN RIEMANNIAN SPACE

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1. Simple proof of a fundamental theorem in a space of positive curvature. Myers[1] has recently shown that a complete Riemannian manifold V_N, all of whose Riemannian curvatures are greater than a positive constant K_0, is closed and has a diameter less than $\pi K_0^{-1/2}$. The fundamental lemma on which this conclusion is based is the following:

THEOREM I. *If all Riemannian curvatures of V_N are greater than or equal to a positive constant K_0, no geodesic arc of length greater than $\pi K_0^{-1/2}$ is the shortest curve joining the end points.*

In Myers' paper, and throughout the present paper, the line-element

$$ds^2 = a_{ij}\, dx^i\, dx^j \tag{1.1}$$

is assumed to be positive definite.

Myers' proof of Theorem I is similar to that given by Schoenberg,[2] and depends on the theory of conjugate points in the N-dimensional sense. However, the theorem is an immediate consequence of a result established by me,[3] which involves only the simpler concept of 2-dimensional conjugate points. But indeed the introduction of the idea of conjugate points and proofs of their existence are quite unnecessary for the establishment of Theorem I, as will now be shown.

Let AB be a geodesic arc of length L. Applying any infinitesimal variation η^i which vanishes at the end points, the first variation is of course zero, and the second variation is[4]

$$\delta^2 L = \frac{1}{2}\int_0^L [\eta'^2 + (\bar{\mu}^2 - K)\eta^2] ds\,, \tag{1.2}$$

where η is the magnitude of η^i, $\eta' = d\eta/ds$, $\bar{\mu}$ is the magnitude of the absolute derivative of the unit vector μ^i co-directional with η^i, and K is the Riemannian curvature of V_N for the 2-element containing the tangent to the geodesic and η^i. We may choose the unit vector μ^i as we like along AB, and we can assign η as we like, provided that $\eta = 0$ at A and at B. Let μ^i be propagated parallelly, so that $\bar{\mu} = 0$. Then, since by hypothesis $K \geqq K_0$, we have

$$\delta^2 L = \frac{1}{2}\int_0^L (\eta'^2 - K\eta^2)\, ds \leqq \frac{1}{2}\int_0^L (\eta'^2 - K_0\eta^2)\, ds\,. \tag{1.3}$$

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[1] S. B. Myers, this Journal, vol. 1 (1935), p. 42.

[2] I. J. Schoenberg, Annals of Math., vol. 33 (1932), p. 493.

[3] J. L. Synge, Proc. London Math. Soc., vol. 25 (1925), p. 264, Theorem XVII.

[4] Ref. 3, p. 261, equation (9.17); the notation is slightly changed.