ON THE NUMBER THEORY OF CERTAIN NON-MAXIMAL DOMAINS OF THE TOTAL MATRIC ALGEBRA OF ORDER 4

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1. **Introduction.** This paper is devoted to the investigation of the number theory of certain non-maximal domains of integrity of the total matric algebra of order 4.

We shall call a *domain of integrity* (or merely a *domain*) of the above algebra any subset which (1) is of order 4, (2) contains the identity matrix and (3) is closed under addition and multiplication, the constants of multiplication being rational integers.

A[1] canonical basis has been derived for such domains under certain transformations. We shall make a study of a subset of these domains, obtaining some interesting properties. If we take the basis mentioned above under case I and set $m = l = 0$, $a = 1$ and let k be a prime, we get a basis which is evidently equivalent to

$$(1) \qquad \begin{Vmatrix} 1 & 0 \\ 0 & 0 \end{Vmatrix}, \quad \begin{Vmatrix} 0 & 1 \\ 0 & 0 \end{Vmatrix}, \quad \begin{Vmatrix} 0 & 0 \\ 0 & 1 \end{Vmatrix}, \quad \begin{Vmatrix} 0 & 0 \\ k & 0 \end{Vmatrix}.$$

We shall refer to the above matrices as E_1, E_2, E_3, and E_4 in the order given. If $k = 1$, we get the unique maximal domain of the algebra. In this paper $|k| > 1$. We shall refer to (1) as the domain D.

In paragraph 2 we obtain a set of canonical forms for the numbers of the domain under consideration. From this it follows that a necessary and sufficient condition that a number be indecomposable in the domain is that its determinant be a rational prime. Hence we have an example of a simple non-commutative domain of class number greater than one for which the indecomposable numbers are known.

Paragraph 3 is devoted to the determination of the class number of the domain. I believe this is the first determination of the class number of a non-commutative domain with class number greater than 1—that of quaternions being 1. This approach to the theory of ideals through matrices with rational integral elements is the same as that used by C. C. MacDuffee.[2]

2. **Canonical forms for numbers of the domain.** Any number in the domain D may be written in the form $N = \Sigma n_i E_i$, where the n_i are rational integers. Evidently such a number is a rational integral square matrix of

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[1] E. J. Finan, American Journal of Mathematics, vol. 54 (1931), pp. 920–928.

[2] C. C. MacDuffee, Transactions of the Amer. Math. Soc., vol. 31 (1929), pp. 71–90.