ON CRITICAL SETS

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A number of topological investigations of recent years have had for objective the freeing of algebraic topology from extraneous considerations and hypotheses. The most noteworthy are perhaps the endeavors to obtain an intrinsic theory of manifolds. The starting point has been to replace the regularity typified by euclidean spaces by conditions expressed exclusively in terms of chains and homologies. Our present purpose is to exploit our recent results on chain-deformations[1] to bring Morse's important theory of critical points[2] within the framework of algebraic topology. The treatment will be found to be substantially free from the difficulties usually connected with homotopy.

That the free use of topology more than justifies itself will immediately be perceived from the examples given in No. 2 and from the treatment of critical points of functions on manifolds which occupies the last section of the paper.

§1. Critical values of a function on a set

1. Let $f(x)$ be a real continuous function of the point x on a metric space $\mathfrak{R}$. We propose to investigate the variation of the homology structure of the spaces $a \leqq f < y$, $a \leqq f \leqq y$ as y varies. The range taken at any one time shall be from some finite a up, although the restriction is relatively unimportant. In fact, the range might well be the whole real axis or the real numbers mod 1 (circumference) without entailing essential modifications in the treatment.

Upon examination, Morse's theory is found to demand essentially (a) chain-deformation downwards across all but some isolated sets $y =$ constant, and also downwards away from all these sets without exception; (b) the finiteness of the homology characters (type numbers). Our structural axioms practically amount to imposing these properties.

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[1] See S. Lefschetz, *Chain-deformations in topology*, Duke Journal, vol. 1 (1935), pp. 1–19 (= DJ in the sequel). The assumptions, notations and mode of reference of DJ will be used in the present paper. In particular, *Topology* designates our Colloquium Lectures. We also recall that the symbols HLC, HNR stand, respectively, for "locally connected" and "neighborhood retract" in the sense of homology.

[2] The chief references (prior to 1934) to the very extensive literature, chiefly due to Morse and his students, which has grown up in recent years around these questions will be found at the end of Morse's Colloquium Lectures, *The Calculus of Variations in the Large*, New York, 1934, referred to as MC in the sequel. In addition, we may mention our recent note in the Proceedings of the National Academy, and a subsequent note by Morse-Van Schaack, ibid., pp. 258–263.