CONNECTIONS BETWEEN DIFFERENTIAL GEOMETRY AND TOPOLOGY

I. SIMPLY CONNECTED SURFACES

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Introduction. In this paper is presented a theory of new connections between differential geometry and topology. With an arbitrary point A of a complete analytic Riemannian surface S is associated a locus of "minimum points with respect to A". A point M on a geodesic ray g issuing from A is said to be a "minimum point with respect to A on g" if M is the last point on g such that AM furnishes an absolute minimum to the arc length of curves on S joining A to M. The locus of such points with respect to A is proved to be a linear graph m. If S is simply connected, m is a tree when S is closed and a set of infinite trees when S is open. In a later paper, it will be proved that in the general case of a closed multiply connected S, m is a linear graph whose cyclomatic number is equal to the connectivity number modulo 2 of S. The surface S is thus reduced to a single 2-cell σ with m as its singular boundary; σ is simply covered (except at A) by the geodesic rays through A cut off at their intersections with m and hence can be represented by the geodesic polar coördinate system with A as pole. This solves completely the hitherto vaguely answered question as to how long the geodesics through a point A of a surface form a field.

This paper is restricted mainly to simply connected surfaces, hence surfaces homeomorphic to the plane or the sphere. The end points of the branches of the tree (or trees) which forms the locus of minimum points with respect to an arbitrary point A on S are shown to be conjugate to A and to be cusps of the locus of first conjugate points to A. The order of a point M of m as a vertex of m (i.e., the number of arcs of m issuing from M) is proved to be equal to the number of geodesics joining M to A on which M is a minimum point with respect to A.

In order to prove that in the case of a closed analytic simply connected surface the locus m has a finite number of end points, it is necessary to study the locus of first conjugate points to A. This is done in §3. §1 recalls the definition of a complete analytic Riemannian surface, while in §2 the machinery is set up for finding the conjugate point locus. In §4 the minimum point locus is studied. In §5 examples are given of the minimum point locus on a few simple surfaces, while in §6 problems arising from the methods and results of this paper are suggested and possibilities of generalization discussed.

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