ASYMPTOTIC AND PRINCIPAL DIRECTIONS AT A PLANAR POINT OF A SURFACE

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1. Introduction. If a regular point P of a real, analytic surface

$$S: \qquad x = x(u, v), \qquad y = y(u, v), \qquad z = z(u, v),$$

is not an umbilic, there exist at P two pairs of directions, the *asymptotic* and the *principal* directions, defined respectively by the equations

$$e\,du^2 + 2f du dv + g\,dv^2 = 0, \tag{1}$$

$$\begin{vmatrix} e\,du + f\,dv & E\,du + F\,dv \\ f\,du + g\,dv & F\,du + G\,dv \end{vmatrix} = 0, \tag{2}$$

where E, F, G and e, f, g, the coefficients in the first and second fundamental forms of S, are evaluated at P. These directions have the following well-known properties.

1. The tangent plane to S at P has contact of at least the second order in the asymptotic directions.
2. The normal curvature of S at P vanishes in the asymptotic directions and has its extrema in the principal directions.
3. The geodesic torsion of S at P vanishes in the principal directions.
4. The asymptotic lines are the solutions, for variable u and v, of the differential equation (1). Through each regular non-planar point there pass, in the asymptotic directions, two asymptotic lines.
5. The lines of curvature are the solutions of (2). Through each regular point which is not an umbilic there pass, in the principal directions, two real, distinct lines of curvature.

A *planar point* is defined as a regular point at which the quantities e, f, g vanish simultaneously. At a planar point the normal curvature and the geodesic torsion of the surface are zero in all directions, so that the first three theorems above become trivial. The fourth and fifth are false, because the point is singular for the differential equations (1) and (2).

In this paper there are defined two sets of directions at a planar point which play rôles similar to those played in the first three theorems by the asymptotic and principal directions at an ordinary point. These directions will be called the "true asymptotic" and the "true principal" directions. In a second paper the analogues of the fourth and fifth theorems will be established by studying the solutions at a planar point of the differential equations (1) and (2).

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