NATURAL ISOPERIMETRIC CONDITIONS IN THE CALCULUS OF VARIATIONS

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Introduction

In the ordinary problems of the Calculus of Variations, an extremal is defined in general as an arc along which the first variation of the prescribed integral J vanishes. Such an arc is completely characterized by the fact that it satisfies the attached Euler differential equations and transversality conditions.

Among such extremal arcs there are certain ones which furnish a minimum value of J under the prescribed boundary conditions. It has always been a matter of primary theoretical interest in the Calculus of Variations to determine when such a minimum is realized. In the case of a minimum the minimizing arc can be found by use of a minimizing principle.

For extremal arcs not furnishing a minimum the attached quadratic accessory minimum problem no longer is analogous to a positive definite quadratic form. It has long been recognized that the number of negative terms, or 'type number', in this normalized form is equal to the number of negative characteristic values in the associated linear self-adjoint boundary value problem. This fact has been especially apparent ever since the formulation of the theory of homogeneous linear integral equations with real symmetric kernels. It is implicitly involved in the work of Birkhoff[1] without regard to the boundary value problem, in particular for the case of type number zero (minimum) and type number one (minimax). Morse[2] has subsequently developed the notion of type number systematically, also using the method of broken extremals.

The question arises as to whether or not all extremals whatsoever may not be obtained as a solution of a properly formulated minimum problem. In the present paper we show that this is indeed the case. More definitely, we show that by adding a suitable set of 'natural isoperimetric conditions', *which are automatically satisfied by every possible extremal fulfilling the given conditions*, there is obtained a related isoperimetric problem for which the extremal arc in question is a minimizing arc.[3]

So far as we know, the only case in which such natural isoperimetric condi-

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[1] See his paper *Dynamical systems with two degrees of freedom*, Trans. Amer. Math. Soc., vol. 18 (1917), in particular, pp. 239–257.

[2] For references to the works of Morse, see his book *The Calculus of Variations in the Large*, Colloquium Publications, American Mathematical Society, vol. 18 (1934). Unless otherwise expressly stated, all references to Morse in the following pages are to his book.

[3] See our paper in the Proceedings of the National Academy of Sciences, February, 1935, with the same title, in which this general principle is first formulated.