LINEAR ALGEBRAS WITH ASSOCIATIVITY NOT ASSUMED

By L. E. Dickson

1. The complete structure of linear associative algebras was known to depend upon the division algebras. When the reference field F is an algebraic field, H. Hasse has recently proved that every normal[1] division algebra is cyclic. This perfection of the theory of associative algebras justifies attention to non-associative algebras.

Known examples of non-associative division algebras are Cayley's algebra of order 8, and the writer's[2] commutative algebras of orders 3 and $2n$ (§15). Many new division algebras of order 4 are given here by Theorems 2 and 3.

In §§7–11 we determine all types of algebras of order 3 having a principal unit (or modulus) denoted by 1. Except for special values of the parameters, these algebras are simple. It is known that every associative simple algebra of order 3 is a division algebra.

Thus the structure theorems for associative algebras fail in general for non-associative algebras. Similarly for other properties. Consider the algebra A of order 4 with $e_1^2 = e_2$, $e_2 e_1 = e_3$, and all further $e_i e_j = 0$. For $X = x_0 + \Sigma x_i e_i$, evidently $(X - x_0)(X - x_0)^2 = 0$, so that A has the left rank 3. But its right rank is 4 since $1, e_1, e_1^2 = e_2, e_1^2\, e_1 = e_3$ are linearly independent. Algebra (37) with 13 parameters also has left and right ranks 3 and 4.

Part I. Rank 2

2. In case the field F has a modulus p, assume that $p \neq 2$.

Lemma 1. *If* $1, u, v$ *are linearly independent with respect to* F, *and* $u^2 = a + cu$, $v^2 = b + dv$, *then*

$$uv + vu = du + cv + f, \qquad f \text{ in } F.$$

Write t for $a + b + cu + dv$. Then

$$(u + v)^2 = t + uv + vu = r + s(u + v),$$

$$(u - v)^2 = t - uv - vu = R + S(u - v).$$

Addition yields $s + S = 2c$, $s - S = 2d$. Subtraction gives Lemma 1.

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[1] If only the numbers of F are commutative with every element.

[2] *Linear Algebras*, Cambridge Tract No. 16, p. 14, p. 69 (p. 17 for the characteristic equations); *On triple algebras and ternary cubic forms*, Bull. Amer. Math. Soc., vol. 14 (1907–8), pp. 160–169, p. 169; *Linear algebras in which division is always uniquely possible*, Trans. Amer. Math. Soc., vol. 7 (1906), pp. 370–390; *On commutative linear algebras in which division is always uniquely possible*, ibid., pp. 514–522.