Vol. 105, No. 3 DUKE MATHEMATICAL JOURNAL © 2000

GROUP SCHEMES AND LOCAL DENSITIES

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1. Introduction. The subject matter of this paper is an old one with a rich history, beginning with the work of Gauss and Eisenstein, maturing at the hands of Smith and Minkowski, and culminating in the fundamental results of Siegel. More precisely, if L is a lattice over $\mathbb{Z}$ (for simplicity), equipped with an integral quadratic form Q, the celebrated Smith-Minkowski-Siegel mass formula expresses the total mass of (L, Q), which is a weighted class number of the genus of (L, Q), as a product of local factors. These local factors are known as the local densities of (L, Q). Subsequent work of Kneser, Tamagawa, and Weil resulted in an elegant formulation of the subject in terms of Tamagawa measures. In particular, the local density at a non-Archimedean place p can be expressed as the integral of a certain volume form ω^{ld} over $\mathrm{Aut}_{\mathbb{Z}_p}(L, Q)$, which is an open compact subgroup of $\mathrm{Aut}_{\mathbb{Q}_p}(L, Q)$.

The question that remains is whether one can find an explicit formula for the local density. Through the work of Pall (for $p \neq 2$) and Watson (for $p = 2$), such an explicit formula for the local density is in fact known for an arbitrary lattice over $\mathbb{Z}_p$ (see [P] and [Wa]). The formula is obviously structured, although [CS] seems to be the first to comment on this. Unfortunately, the known proof (as given in [P] and [K]) does not explain this structure and involves complicated recursions. On the other hand, Conway and Sloane [CS, Section 13] have given a heuristic explanation of the formula.

In this paper, we give a simple and conceptual proof of the local density formula for $p \neq 2$. The viewpoint taken here is similar to that of our earlier work [GHY], and the proof is based on the observation that there exists a *smooth* affine group scheme $\underline{G}$ over $\mathbb{Z}_p$ with generic fiber $\mathrm{Aut}_{\mathbb{Q}_p}(L, Q)$, which satisfies $\underline{G}(\mathbb{Z}_p) = \mathrm{Aut}_{\mathbb{Z}_p}(L, Q)$. This follows from general results of smoothening [BLR], as we explain in Section 3. To obtain an explicit formula, it is necessary to have an explicit construction of $\underline{G}$. The main contribution of this paper is to give such an explicit construction of $\underline{G}$ (in Section 5), and to determine its special fiber (in Section 6). Finally, by comparing ω^{ld} and the canonical volume form ω^{can} of $\underline{G}$, we obtain the explicit formula for the local density in Section 7. The smooth group schemes constructed in this paper should also be of independent interest.

Our method works over any non-Archimedean local field of residue characteristic $p \neq 2$ and also works for all types of classical groups. Therefore, we obtain new explicit formulas for local densities of lattices in symplectic spaces, Hermitian spaces, and quaternionic Hermitian and anti-Hermitian spaces. For lattices in a symplectic

Received 29 September 1999. Revision received 10 February 2000.
2000 *Mathematics Subject Classification*. Primary 11E41, 11E95, 14L15, 20G25, 20G30; Secondary 11E12, 11E57.