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SCHRÖDINGER OPERATORS WITH DECAYING POTENTIALS: SOME COUNTEREXAMPLES

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1. Introduction. In this paper, we are interested in one-dimensional Schrödinger equations,

$$-y''(x)+V(x)y(x)=Ey(x), \tag{1}$$

with potentials V bounded by a decaying power:

$$|V(x)| \le \frac{C}{(1+x)^{\alpha}} \quad (\alpha > 0). \tag{2}$$

More specifically, we study the spectral properties of the associated selfadjoint operators $H_\beta = -d^2/dx^2 + V(x)$, acting on the Hilbert space $L_2(0,\infty)$. The index $\beta \in [0,\pi)$ refers to the boundary condition $y(0)\cos\beta + y'(0)\sin\beta = 0$. For the general theory, see, for example, [29]. These questions are of relevance in quantum mechanics; for more background information on this topic, refer to, for example, [18].

Since V tends to zero as $x \to \infty$, it follows that the essential spectrum satisfies $\sigma_{\text{ess}} = [0,\infty)$. (This is a classical result going back to Weyl.) However, this does not say much about the physics of the corresponding system since it does not give information on the type of the spectrum on $(0,\infty)$ (absolutely continuous, singular continuous, or point spectrum). There has been some progress on this question recently, and several new positive results have been obtained. In this paper, we launch the counterattack: by constructing suitable examples, we show that these results are in fact optimal.

We are interested in situations where $\sigma_{ac} = [0,\infty)$ with possibly also some embedded singular spectrum on $(0,\infty)$. The corresponding range of exponents in (2) is $1/2 < \alpha \le 1$. If $\alpha > 1$ or if only $V(x) = o(x^{-1})$, then the spectrum is purely absolutely continuous on $(0,\infty)$. See [20] for the proof of this under the weaker assumption $V(x) = o(x^{-1})$; if $\alpha > 1$ is assumed, the result is classical and easy to prove. On the other hand, there are examples of (random) potentials $V(x) = O(x^{-1/2})$ with purely singular spectrum (see [6] and [24]), so there need not be any absolutely continuous spectrum if $\alpha \le 1/2$. That $\alpha > 1/2$ does imply presence of absolutely

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