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SECOND-ORDER DERIVATIVES AND REARRANGEMENTS

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1. Introduction. Let $l > 0$ and $1 \leq p \leq \infty$. Given any nonnegative function u from the Sobolev space $W^{1,p}(0,l)$, the decreasing rearrangement u^* of u is also in $W^{1,p}(0,l)$ and

$$\left\| u^{*\prime} \right\|_{L^p(0,l)} \leq \|u'\|_{L^p(0,l)}. \tag{1.1}$$

Here, prime stands for derivative. The above statement is a variant of the so-called Pólya-Szegö principle. In its original (one-dimensional) formulation, such a principle tells us that for every nonnegative function $u \in W^{1,p}(0,l)$ vanishing at zero and l, the symmetric rearrangement $u^\bigstar$ of u is in $W^{1,p}(0,l)$ and

$$\left\| u^{\bigstar\prime} \right\|_{L^p(0,l)} \leq \|u'\|_{L^p(0,l)}. \tag{1.2}$$

Inequality (1.2) is classical and very well known. It has been the object of extensions and variants which can be found in a number of papers and monographs, including [AFLT], [Bae], [BBMP], [BH], [Bro], [BZ], [CP], [E], [H], [Ka2], [Kl], [Ma], [Sp1], [Sp2], [Spi], [Ta1], [Ta5], and [Ta6]. Even if not as popular as (1.2), inequality (1.1) has also been known for a long time, and versions of it appear in [Ci1], [Du], [Ga], [Ka1], [Ma], [RT], and [Ry]. Various are the applications of Pólya-Szegö-type inequalities. In particular, they have proved to be crucial for such results as isoperimetric inequalities of mathematical physics (see [PS] and [Ta4]) and Sobolev-type inequalities in optimal form (see, e.g., [AFT], [A], [Ci2], [CF], [EKP], [L], [Ko], [Mo], [Ta1], and [Ta5]). They are also strictly connected to a priori sharp estimates for solutions to second-order elliptic and parabolic boundary value problems (see [ALT], [Ba], [Di], [Ka1], [Ke], [Ta2], and [Ta3]).

On the other hand, the effect of rearrangements on Dirichlet-type functionals depending on higher-order derivatives seems to be still unknown. The present paper is aimed at giving a contribution on this subject. Indeed, we are concerned with inequalities in the spirit of (1.1) and (1.2) involving second-order derivatives. An evident obstacle in attacking this question is that very smooth functions may have a decreasing (and symmetric) rearrangement whose first-order derivative is not even weakly differentiable (see Section 2, Remark 3, for an example). Thus, unlike $W^{1,p}(0,l)$, membership of a function in the second-order Sobolev space $W^{2,p}(0,l)$ need not be preserved after rearranging it in decreasing order. This shortcoming can be overcome by enlarging the class of admissible functions. Actually, our main result—Section 2,

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