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REMARKS ON QUIVER VARIETIES

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0. Introduction

0.1. In several respects, the quiver varieties introduced by Nakajima [N1] are analogous to the variety $\mathcal{B}_e$ of Borel subalgebras in a complex simple Lie algebra containing a given nilpotent element e. It would be very interesting to define a canonical basis of the equivariant K-theory of a quiver variety analogous to the (conjectural) one proposed in [L1] and [L3] for $\mathcal{B}_e$. (This would give something very close to the canonical basis of the modified quantized affine enveloping algebra, whose geometric interpretation has been elusive until now.) In this paper, we take (what we hope is) a small step in this direction. All the ingredients used for $\mathcal{B}_e$ (except one) make sense for a quiver variety, and here we concentrate on finding the missing ingredient, the analogue of an opposition involution. For this purpose, it is essential to use the model of a quiver variety introduced in [L2], not as an orbit space (as in [N1]), but as a kind of Grassmannian. More precisely, it appears as the variety of submodules of a projective module (of finite-dimension over $\mathbf{C}$) over the Gelfand-Ponomarev algebra [GP] corresponding to a Coxeter graph of type A, D, and E.

In §1 we show that the duals of these projective modules are again projective, and the resulting duality allows us to construct an analogue of the opposition involution. In §2 we speculate on how the canonical basis may be defined in our case. In §3 we study a partition of a quiver variety.

1. New symmetries of quiver varieties

1.1. Assume that we are given two finite sets I, H with $I \neq \emptyset$, two maps $H \to I$ denoted as $h \mapsto h'$ and $h \mapsto h''$ such that $h' \neq h''$ for all $h \in H$, and an involution $h \mapsto \bar{h}$ of H such that $(\bar{h})' = h''$ for all $h \in H$. We say that $i, j \in I$ are joined if there exists $h \in H$ such that $h' = i$, $h'' = j$. Thus I becomes the set of vertices of a graph.

The path algebra $\mathcal{F}$ associated to (I, H) is the associative $\mathbf{C}$-algebra with 1 defined by the generators e_i $(i \in I)$, h $(h \in H)$ and the relations

$e_i e_j = \delta_{i,j} e_i$ for $i, j \in I$,

$\sum_i e_i = 1$,

$e_{h''} h = h = h e_{h'}$ for $h \in H$.

It follows that $h_1 h_2 = 0$ for $h_1, h_2 \in H$ such that $h_1' \neq h_2''$.

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