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THE RAMANUJAN PROPERTY FOR REGULAR CUBICAL COMPLEXES

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0. Introduction. Ramanujan graphs were defined by Lubotzky, Phillips, and Sarnak in [15] as regular graphs whose adjacency matrices, or their Laplacians, have eigenvalues satisfying some "best possible" bounds. Such graphs possess many interesting properties. In this paper, we give a higher-dimensional generalization of this theory to *regular cubical complexes*. By definition, ($\vec{r} = (r_1, \dots, r_g)$)-regular complexes are cell complexes locally isomorphic to the (ordered) product of g regular trees, with the jth tree of regularity $r_j \geq 3$. Each cell is an i-cube (i.e., an i-dimensional cube) with $0 \leq i \leq g$. Throughout each $(g-1)$-cube, exactly one of the tree factors, say the jth one, is constant, and there are r_j g-cubes passing through it. When $g = 1$, we simply have an r-regular graph.

The spaces of i-cochains $C^i(X)$ (with real or complex coefficients) of a finite cubical complex X are inner product vector spaces with an orthonormal basis corresponding to the characteristic functions of the i-cells. There are partial boundary operators $\partial_j = \partial_{j,i} : C^i(X) \to C^{i+1}(X)$ for $1 \leq j \leq g$. With these we get the adjoint operators $\partial_j^* = \partial_{j,i}^* : C^{i+1}(X) \to C^i(X)$ and, hence, the partial Laplacians

$$\square_j = \square_{j,i} = \partial_{j,i}^* \partial_{j,i} + \partial_{j,i-1} \partial_{j,i-1}^* : C^i(X) \longrightarrow C^i(X).$$

Each $\square_{j,i}$ is a selfadjoint nonnegative operator. For i fixed they all commute and one gets a combinatorial harmonic theory (cf. [21]).

When X is infinite, these notions extend to L_2-cochains. When $X = \Delta$ is an $\vec{r} = (r_1, \dots, r_g)$-regular product of trees, Kesten's 1-dimensional results (see [13]) extend, and we get that each λ in the spectrum of $r_j \operatorname{Id} - \square_j$ acting on L_2-cochains of Δ satisfies $|\lambda| \leq 2\sqrt{r_j - 1}$. As in the 1-dimensional case, we say that a $(r_1, \dots, r_g)$-regular cubical complex X is Ramanujan if the eigenvalues of $r_j \operatorname{Id} - \square_j$ on X are $\pm r_j$ or if they satisfy the same properties for each j.

One justification for this definition in the 1-dimensional case is the Alon-Boppana result, which shows that these bounds are essentially the best possible ones for the trivial local system. We generalize this result under a natural hypothesis. Another parallel with the 1-dimensional case is that when X is finite, connected, and uniformized

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