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OSCILLATION AND VARIATION FOR THE HILBERT TRANSFORM

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1. Introduction. For each $\epsilon > 0$, let

$$H_\epsilon f(x) = \frac{1}{\pi}\int_{|t|>\epsilon} \frac{f(x-t)}{t}\,dt.$$

The Hilbert transform, $Hf(x)$, is defined by

$$Hf(x) = \lim_{\epsilon\to 0^+} H_\epsilon f(x).$$

It is well known that this limit exists a.e. for all $f \in L^p$, $1 \le p < \infty$. In this paper, we will consider the oscillation and variation of this family of operators as ϵ goes to zero, which gives extra information on their convergence as well as an estimate on the number of λ-jumps they can have. For earlier results on oscillation and variation operators in analysis and ergodic theory, including some historical remarks and applications, the reader may look in [2], [3], [5], [4], and [6].

For each fixed sequence $(t_i) \searrow 0$, we define the oscillation operator

$$\mathcal{O}\big(H_* f\big)(x) = \left(\sum_{i=1}^{\infty} \sup_{t_{i+1}\le\epsilon_{i+1}<\epsilon_i\le t_i} \big|H_{\epsilon_i} f(x) - H_{\epsilon_{i+1}} f(x)\big|^2\right)^{1/2}$$

and the variation operator

$$\mathcal{V}_\varrho\big(H_* f\big)(x) = \sup_{(\epsilon_i)\searrow 0}\left(\sum_{i=1}^{\infty} \big|H_{\epsilon_i} f(x) - H_{\epsilon_{i+1}} f(x)\big|^\varrho\right)^{1/\varrho}.$$

The main results of this paper are the following two theorems.

THEOREM 1.1. *The oscillation operator* $\mathcal{O}(H_* f)(x)$ *satisfies* $\|\mathcal{O}(H_* f)\|_p \le c_p\|f\|_p$ *for* $1 < p < \infty$ *and* $m\{x : \mathcal{O}(H_* f)(x) > \lambda\} \le (c/\lambda)\|f\|_1$.

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