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FIBER BROWNIAN MOTION AND THE "HOT SPOTS" PROBLEM

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1. Introduction. The main purpose of this article is to give a stronger counterexample to the "hot spots" conjecture than the one presented in Burdzy and Werner [7]. Along the way we define and partly analyze a new process, which we call *fiber Brownian motion*, and which may have some interest of its own.

Consider a Euclidean domain that has a discrete spectrum for the Laplacian with Neumann boundary conditions, for example, a bounded domain with Lipschitz boundary. Recall that the first Neumann eigenfunction is constant. The hot spots conjecture says that the maximum of the second Neumann eigenfunction is attained at a boundary point. Burdzy and Werner [7] constructed a domain where the second eigenfunction attains its maximum inside the domain but its minimum lies on the boundary. If φ is an eigenfunction, so is $-\varphi$, and hence the maximum and minimum are indistinguishable in the context of this problem. Hence the counterexample of [7] leaves open the following question.

Question 1.1. Must at least one of the extrema of the second Neumann eigenfunction be attained on the boundary of the domain?

The uncertainty about the answer to this question is underscored by the nature of Burdzy and Werner's counterexample, which cannot be easily modified to solve Question 1.1. The second author learned (in a private communication) about a different counterexample, obtained by D. Jerison and N. Nadirashvili, which seems to have the same property—that the minimum of the eigenfunction lies on the boundary. The answer to Question 1.1 is given in the following statement.

THEOREM 1.2. *There exists a bounded Lipschitz domain in the plane such that its second Neumann eigenvalue is simple, and both extrema of the corresponding eigenfunction are attained only at interior points of the domain.*

We believe that each extremum of the eigenfunction in Theorem 1.2 is attained at a single point, but we do not prove this. Our example is based on an idea of W. Werner's, involving two triangles whose vertices are connected by very thin tubes.

The hot spots conjecture was proposed by J. Rauch at a conference in 1974. The only published statement of the conjecture is contained in a book by B. Kawohl [11].

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