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EXTREMAL HERMITIAN METRICS ON RIEMANN SURFACES WITH SINGULARITIES

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0. Introduction. It is a well-known consequence of the classical uniformization theorem that there is a metric with constant Gaussian curvature in each conformal class of any compact Riemann surface. It is natural to ask how to generalize this classical uniformization theory to compact surfaces with conical singularities and with nonempty boundary, or to find a "best metric" on such surfaces. However, there are surfaces with conical singularities that do not admit a metric with constant curvature. For example, a football with two different singular angles does not admit a metric with constant curvature. (For the existence or nonexistence results of constant curvature metric in a surface with conical singularities, see [T], [M], [CL2], [CY], [LT], and [UY].) Recently, instead of using metrics of constant curvature, in [Ch5], [Ch4], X. X. Chen started to use the extremal Hermitian metrics to generalize the classical uniformization theory to Riemann surfaces with finite conical singularities. On any football there is at least an extremal Hermitian metric. It was claimed in [Ch4], [Ch2] that there is at least an extremal Hermitian metric on any surface with boundary.

This problem may be regarded as the simplest nontrivial case of Calabi's extremal metrics on Kähler manifolds (cf. [Ca1] and [Ca2]).

Let M be a compact Riemann surface with nonempty boundary ∂M. For any Hermitian metric g_0 on M, consider the set $\mathcal{G}(M)$ of metrics with the same area that are pointwise conformal to g_0 and agree with g_0 in a small neighborhood of ∂M. In the closure of this set $\mathcal{G}(M)$ under some suitable $H^{2,2}$-norm, we define the energy functional

$$E(g) = \int_M K_g^2 \, dg, \tag{0.1}$$

where K_g is the Gaussian curvature of g. A critical point of this functional is called an extremal Hermitian metric. It is easy to see that the Euler-Lagrange equation of the energy functional is

$$\triangle_g K_g + K_g^2 = c \tag{0.2}$$

for some constant c. Let $g = e^{2\psi}|dz|^2$ be written in local coordinates. Chen [Ch5] observed that (0.2) is equivalent to

$$K_{,zz\bar{z}} = 0, \tag{0.3}$$

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