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# A REMARK ON THE ENERGY BLOW-UP BEHAVIOR FOR NONLINEAR HEAT EQUATIONS

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**1. Introduction.** We are concerned with finite-time blow-up for the following nonlinear heat equation:

$$\begin{cases} u_t = \Delta u + |u|^{p-1}u & \text{in } \Omega \times [0,T), \\ u = 0 & \text{on } \partial\Omega \times [0,T) \end{cases} \tag{1}$$

with $u(x,0) = u_0(x)$, where $u : \Omega \times [0,T) \to \mathbb{R}$, $\Omega$ is a $C^{2,\alpha}$-convex bounded domain of $\mathbb{R}^N$, $u_0 \in L^\infty(\Omega)$. We assume that the following condition holds:

$$1 < p, \qquad (N-2)p < N+2, \qquad \text{and} \qquad \left(u_0 \geq 0 \text{ or } p < \frac{3N+8}{3N-4}\right). \tag{2}$$

Therefore, $p+1 > N(p-1)/2$ and the (local in time) Cauchy problem for (1) can be solved in $L^{p+1}(\Omega)$ (see, for instance, [21, Theorem 3]). If the maximum existence time $T > 0$ is finite, then $u(t)$ is said to blow up in finite time, and in this case

$$\lim_{t\to T} \|u(t)\|_{L^{p+1}(\Omega)} = \lim_{t\to T} \|u(t)\|_{L^\infty(\Omega)} = +\infty \tag{3}$$

(see [21, Corollary 3.2]). We consider such a blow-up solution $u(t)$ in the following.

From the regularizing effect of the Laplacian, $u(t) \in L^\infty \cap H_0^1(\Omega)$ for all $t \in (0,T)$. We take $\|u\|^2_{H_0^1(\Omega)} = \int_\Omega |\nabla u|^2\,dx$. Using the Sobolev embedding and the fact that $p$ is subcritical ($p < (N+2)/(N-2)$ if $N \geq 3$), we see that $H_0^1(\Omega) \subset L^{p+1}(\Omega)$. Therefore, (3) implies that

$$\lim_{t\to T} \|u(t)\|_{H_0^1(\Omega)} = +\infty.$$

A point $a \in \Omega$ is called a blow-up point of $u$ if there exists $(a_n, t_n) \to (a, T)$ such that $|u(a_n, t_n)| \to +\infty$.

The set of all blow-up points of $u(t)$ is called the blow-up set and denoted by $S$. From Giga and Kohn [8, Theorem 5.3], there are no blow-up points in $\partial\Omega$. Therefore, we see from (3) and the boundedness of $\Omega$ that $S$ is not empty.

Many papers are concerned with the Cauchy problem for (1) (see, for instance, [21]) or the problem of finding sufficient blow-up conditions on the initial data (see

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