ON THE CHERN-GAUSS-BONNET INTEGRAL FOR CONFORMAL METRICS ON R^4

SUN-YUNG A. CHANG, JIE QING, AND PAUL C. YANG

1. Introduction. An important landmark in the theory of surfaces is the introduction of the notion of complete open surface by Hopf and Rinow [HR]. Subsequently, Cohn-Vossen [CV] studied the Gauss-Bonnet integral for such a surface M with analytic metrics. Cohn-Vossen also showed that if the Gaussian curvature K is absolutely integrable, then

$$\int_M K\,dv_M \leq 2\pi\chi(M), \tag{1.1}$$

where $\chi(M)$ is the Euler number of M. Later, Huber [Hu] extended this inequality to metrics with much weaker regularity. More importantly, he proved that such a surface M is conformally equivalent to a closed surface with finitely many punctures.

The deficit in formula (1.1) has an interpretation as an isoperimetric constant. On a complete and open surface with Gaussian curvature absolutely integrable, one may represent each end conformally as $R^2 \setminus K$ for some compact set K. We consider the isoperimetric ratio

$$\nu = \lim_{r\to\infty} \frac{L^2(r)}{4\pi A(r)},$$

where $L(r)$ is the length of the boundary circle $\partial B_r = \{|x| = r\}$, and $A(r)$ is the area of the annular region $B(r)\setminus K$. For a fairly large class of complete surfaces, Finn [F] showed that

$$\chi(M) - \frac{1}{2\pi}\int_M K\,dv_M = \sum \nu_j, \tag{1.2}$$

where the sum is taken over each end of M. For more recent development on the subject of complete surface of finite total curvature the reader is referred to the work of Li and Tam [LT]. Except for the work of Cheeger and Gromov [CG] on the Chern-Gauss-Bonnet formula for manifolds with bounded geometry, there is very little known about the situation in higher dimensions.

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