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REMARKS ON A HOPF ALGEBRA FOR DEFINING MOTIVIC COHOMOLOGY

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1. Introduction. Let F be a field. In [1], [3], Beilinson et al. considered the groups A_n of pairs of configuration of hyperplanes, subject to a set of relations, in projective spaces $\mathbb{P}^n(F)$. We call these groups generalized scissors congruence groups (see Definition 2.2). The defining relations essentially reflect the functional equations of Aomoto polylogarithms, which, in turn, are connected with multivalued multiple polylogarithms defined in [5]. They were led to study these groups when they were considering the cohomology $H^n(\mathbb{C}P^n \setminus L, M \setminus L)$ by using Deligne's theory of mixed Hodge structures, where (L, M) is an admissible pair of simplices in $\mathbb{C}P^n$ (see Definition 2.1). From the Hodge-theoretic point of view, they realized that the groups $A_n \otimes \mathbb{Q}$ should have a Hopf algebra structure over $\mathbb{Q}$. Then they studied the degree 1 and 2 parts in detail and related the degree 2 part to the Bloch group.

It is now known that in degree 2 and 3 the generalized scissors congruence groups are intimately related to dilogarithms and trilogarithms essentially through their functional equations (see [6]). These groups are also related to hyperbolic geometry in a recent paper by Goncharov [8]. Historically, the relation between K-theory and scissors congruences in hyperbolic spaces first appeared in the paper [4] by Dupont and Sah.

In this paper, we prove that the comultiplication ν of the Hopf algebra structure on the generic part of the generalized scissors congruence groups is well defined. Then we prove it is coassociative. We do not attempt to solve the problem of defining ν on all of A_n, which is seen to be quite complicated at present (see [9]).

Let A_n^0 be the generic part of the grade n-piece of the generalized scissors congruence groups. The coassociativity of ν is important in that it yields the following $A_\bullet^0$ complex (see Corollary 3.5):

$$A_n^0 \xrightarrow{\nu} \bigoplus_{i=1}^{n-1} A_{n-i}^0 \otimes A_i^0 \xrightarrow{\nu \otimes \mathrm{id} - \mathrm{id} \otimes \nu} \bigoplus_{\substack{k_1+k_2+k_3=n \\ k_1,k_2,k_3 \geq 1}} A_{k_1}^0 \otimes A_{k_2}^0 \otimes A_{k_3}^0 \longrightarrow \cdots,$$

which is conjectured to have the same cohomology as that of the A-complex, though no strong evidence has been found so far.

But one should not only consider the generic part of the generalized scissors congruence groups. Calculations by Goncharov show that some defining relations

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