Vol. 103, No. 2 DUKE MATHEMATICAL JOURNAL © 2000

AFFINE MAPPINGS OF TRANSLATION SURFACES: GEOMETRY AND ARITHMETIC

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1. Introduction. Translation surfaces naturally arise in the study of billiards in rational polygons (see [ZKa]). To any such polygon P, there corresponds a unique translation surface, $S = S(P)$, such that the billiard flow in P is equivalent to the geodesic flow on S (see, e.g., [Gu2], [Gu3]).

There is also a classical relation between translation surfaces and quadratic differentials on a Riemann surface S. Namely, each quadratic differential induces a translation structure on a finite puncturing of S or on a canonical double covering of S. Quadratic differentials have a natural interpretation as cotangent vectors to Teichmüller space, and this connection has proven useful in the study of billiards (see, e.g., [Ma2], [V1]).

With a translation surface, S, one associates various algebraic and geometric objects: the induced affine structure of S and the group of affine diffeomorphisms, $\mathrm{Aff}(S)$; the holonomy homomorphism, $\mathrm{hol} : \pi_1(S) \to \mathbf{R}^2$ and the holonomy group $\mathrm{Hol}(S) = \mathrm{hol}(\pi_1(S))$; the flat structure on S and the natural cell decompositions of its metric completion $\overline{S}$. In the present paper, we study the relations between these objects, as well as relations among different translation surfaces.

Our main focus is the group $\mathrm{Aff}(S)$ and the associated group of differentials, $\Gamma(S) \subset \mathrm{SL}(2, \mathbf{R})$. The study of these groups began as part of W. Thurston's classification of surface diffeomorphisms in [Th2]. This study continued with the work of W. Veech in [V1] and [V2]. Veech produced explicit examples of translation surfaces S for which $\Gamma(S)$ is a nonarithmetic lattice. He showed that if $\Gamma(S)$ is a lattice, then the geodesic flow on S exhibits remarkable dynamical properties. For these reasons, we call $\Gamma(S)$ the *Veech group of* S, and if this group is a lattice, then we call S a *Veech surface*.

We now describe the structure of the paper and our main results. In §2, we establish the setting. In particular, we recall the notion of a G-manifold and associated objects: the developing map, the holonomy homomorphism, and the holonomy group. We introduce the notion of the *differential* of a G-map with respect to a normal subgroup $H \subset G$. We also introduce the *spinal triangulation*, one of several cell decompositions canonically associated to a flat surface with cone points.

Received 8 March 1999. Revision received 27 October 1999.
2000 *Mathematics Subject Classification*. Primary 30F60; Secondary 37D40, 53D25, 30F30, 11F72.
Gutkin partially supported by the National Science Foundation grant numbers DMS-9013220, DMS-9400295.
Judge partially supported by a National Science Foundation postdoctoral fellowship.