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ZERO-CYCLES ON HILBERT-BLUMENTHAL SURFACES

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Introduction. The main object of study in this paper with respect to zero-cycles is a special class of Hilbert-Blumenthal surfaces X, which are defined over $\mathbb{Q}$ as smooth compactifications of quasi-projective varieties $S/\mathbb{Q}$, more precisely, of coarse moduli schemes S that represent the moduli stack of polarized abelian surfaces with real multiplication by the ring of integers in a real quadratic field $F = \mathbb{Q}(\sqrt{d})$. We assume that $d = q$ is a prime $\equiv 1(4)$ and that the class number of F is 1. Then $S(\mathbb{C})$, the complex points of S, can be described as $\mathbb{H} \times \mathbb{H} / \mathrm{SL}_2(O_F)$, where $\mathbb{H}$ is the upper half-plane. In the early seventies, Hirzebruch and Zagier [HZ] defined for each integer N a curve T_N on S (called Hirzebruch-Zagier cycles) and showed that their intersection numbers occur as Fourier coefficients of modular forms of level q with Nebentypes ε_q, the quadratic character of $F/\mathbb{Q}$. In this connection with modular forms, Hirzebruch-Zagier cycles reveal very similar properties to Hecke correspondences on the self-product of the modular curve $X_0(q)$. This crucial observation of Hirzebruch and Zagier, together with Tunnell's proof of the Tate conjecture for a product of modular curves, inspired Harder, Langlands, and Rapoport [HLR] to prove the Tate conjecture for divisors on Hilbert-Blumenthal surfaces over abelian number fields. (The proof of the Tate conjecture was then accomplished by Klingenberg [Kl] and Murty and Ramakrishnan [MR] in the general case.)

From the new strategy to study torsion zero-cycles on algebraic surfaces as developed in [LS] and used in [L1], [L2] (compare also [LR]), it is clear that a crucial point is the Tate conjecture in characteristic p at good reduction primes. As one of our main results, we prove the Tate conjecture in characteristic p for primes p that split in F for a certain class of Hilbert-Blumenthal surfaces. For this we recall that to each modular cusp form f of weight 2, level q, and Nebentypes ε_q, that is, $f \in S_2(\Gamma_0(q), \varepsilon_q)$, we associate a Hilbert modular cusp form $\hat{f} \in S_2(\mathrm{SL}_2(O_F))$ under the Doi-Naganuma lift DN_F (compare [vdG]). Then we prove the following theorem.

THEOREM A. *Let p be a prime that splits in F and such that X has good reduction at p. Let X_p be the closed fiber of a smooth, proper model $\mathscr{X}/\mathbb{Z}_p$ of X. Assume that DN_F is surjective. Then the Tate conjecture holds for divisors on X_p.*

Examples of such surfaces can be found in [O].

From the work of [HLR], we know that the interesting part in the second étale cohomology of X is intersection cohomology that decomposes into isotypic components

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