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INTERSECTION COHOMOLOGY OF S^1 SYMPLECTIC QUOTIENTS AND SMALL RESOLUTIONS

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1. Introduction. Let a compact Lie group G act effectively on a compact connected symplectic manifold M with a moment map $\Phi : M \to \mathfrak{g}^*$. When 0 is a regular value of the moment map, the symplectic quotient (reduced space) $M_{\text{red}} := \Phi^{-1}(0)/G$ is an orbifold and its rational cohomology ring is fairly well understood (see [GK], [JK], [Ka], [Ki1], [TW], [Wi], [Wu]).

However, many interesting spaces arise as reduced spaces at singular values of the moment map. Some examples include the moduli space of flat connections, some polygon spaces, many physical systems, and singular projective toric varieties. Since the symplectic quotient at a singular value is a stratified space [SL], a natural invariant to compute is the intersection cohomology (with middle perversity). Less is known in this case. Kirwan has provided formulas to compute the Betti numbers in the algebraic case (see [Ki1], [Ki2], [Ki3]); Woolf extended this work to the symplectic case [Wo]. Moreover, Jeffrey and Kirwan computed the pairing in the intersection cohomology of particular symplectic quotients [Ki5].

The main result of this paper is two explicit formulas for the intersection cohomology (as a graded vector space with pairing) of the symplectic quotient by a circle in terms of the S^1 equivariant cohomology of the original symplectic manifold and the fixed-point data. More precisely, these formulas depend on the image of the restriction map in equivariant cohomology from the original manifold to the fixed-point set, $H^*_{S^1}(M;\mathbb{R}) \to H^*_{S^1}(M^{S^1};\mathbb{R})$. Additionally, we show that the intersection cohomology of the reduced space admits a compatible ring structure.

THEOREM 1. *Let the circle S^1 act on a compact connected symplectic manifold M with moment map $\Phi : M \to \mathbb{R}$ so that 0 is in the interior of $\Phi(M)$. Let $M_{\text{red}} := \Phi^{-1}(0)/S^1$ denote the reduced space.*

There exists a surjective map κ from the equivariant cohomology ring $H^*_{S^1}(M;\mathbb{R})$ to the intersection cohomology $IH^*(M_{\text{red}};\mathbb{R})$. Moreover, given any equivariant cohomology classes α and β in $H^*_{S^1}(M)$, the pairing of $\kappa(\alpha)$ and $\kappa(\beta)$ in $IH^*(M_{\text{red}})$ is given by the formula

$$\langle \kappa(\alpha), \kappa(\beta) \rangle = \operatorname{Res}_0 \sum_{F \in \mathcal{F}^+} \int_F \frac{i_F^*(\alpha\beta)}{e_F}.$$

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