BMO FOR NONDOUBLING MEASURES

J. MATEU, P. MATTILA, A. NICOLAU, AND J. OROBITG

1. Introduction. The Calderón-Zygmund theory of singular integrals has been traditionally considered with respect to a measure satisfying a doubling condition. Recently, Tolsa [T] and, independently, Nazarov, Treil, and Volberg [NTV] have shown that this standard doubling condition was not really necessary. Likewise, in the homogeneous spaces setting, functions of bounded mean oscillation, BMO, and its predual H^1, the atomic Hardy space, play an important role in the theory of singular integrals.

This note is an attempt to find good substitutes for the spaces BMO and H^1 when the underlying measure is nondoubling. Our hope was that we would have been able to prove some results of Tolsa, Nazarov, Treil, and Volberg, via BMO-H^1 interpolation, but in this respect we were unsuccessful.

Let μ be a nonnegative Radon measure on $\mathbb{R}^n$. A function $f \in L^1_{\mathrm{loc}}(\mu)$ is said to belong to $\mathrm{BMO}(\mu)$ if the inequality

$$\int_Q |f(x) - f_Q|\, d\mu(x) \le C\mu(Q) \tag{1}$$

holds for all cubes Q with sides parallel to the coordinate axes; $f_Q = (\mu(Q))^{-1} \int_Q f\, d\mu$ denotes the mean value of f over the cube Q. The smallest bound C for which (1) is satisfied is then taken to be the "norm" of f in this space, and it is denoted by $\|f\|_*$.

One says that $\mathrm{BMO}(\mu)$ has the John-Nirenberg property when there exist positive constants c_1 and c_2 so that whenever $f \in \mathrm{BMO}(\mu)$, then for every $\lambda > 0$ and every cube Q with sides parallel to the coordinate axes, one has

$$\mu\big(\{x \in Q : |f(x) - f_Q| > \lambda\}\big) \le c_1 e^{-c_2\lambda/\|f\|_*}\mu(Q).$$

It is well known that if a measure μ is doubling (i.e., there exists a constant $C = C(\mu)$ such that $\mu(2Q) \le C\mu(Q)$ for all cubes Q), then it satisfies the John-Nirenberg inequality. We give examples of nonnegative Radon measures on $\mathbb{R}^n$ which do not

Received 8 January 1999. Revision received 3 August 1999.

1991 *Mathematics Subject Classification*. Primary 26B35, 42B25, 46E30.

Mateu, Nicolau, and Orobitg's work partially supported by grant numbers PB98-0872 and PB96-1183 of the Dirección General de Enseñanza Superior and by grant number 1998 SGR00052 of the Comissió Interdepartamental de Recerca i Innovació Tecnològica.

Mattila gratefully acknowledges the hospitality of the Centre de Recerca Matemàtica at the Universitat Autònoma de Barcelona during the preparation of this work.