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EXPONENTIAL DECAY IN THE FREQUENCY OF ANALYTIC RANKS OF AUTOMORPHIC L-FUNCTIONS

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This note should be seen as an appendum to the work of E. Kowalski and the second author [KM1] and deals with the problem of bounding, unconditionally, the order of vanishing at the critical point in a family of L-functions. This problem is illustrated in the particular case of L-functions of weight-2 primitive modular forms of prime level. Recall the notation from [KM1]: For q a prime number, let $S_2(q)^*$ be the set of primitive forms of weight 2 and level q, normalized so that their first Fourier coefficient is 1; for

$$f(z) = \sum_{n\geq 1} \lambda_f(n) n^{1/2} \mathrm{e}(nz) \in S_2(q)^*, \quad \lambda_f(1) = 1,$$

let

$$L(f,s) := \sum_{n\geq 1} \frac{\lambda_f(n)}{n^s},$$

the associated (normalized) L-function; it admits analytic continuation to $\mathbf{C}$ with a functional equation relating $L(f,s)$ to $L(f,1-s)$, and we call $r_f := \mathrm{ord}_{s=1/2} L(f,s)$ the analytic rank of f. In [KM1] the following was proved: There exists an absolute constant $C_1 \geq 0$ such that, for all q prime,

$$\sum_{f\in S_2(q)^*} r_f \leq C_1 |S_2(q)^*|.$$

After that, much progress has been made concerning this question. In particular, in [KMV], a sharp explicit value was given for the constant C_1 ($C_1 = 1.1891$ for q large enough). In the course of the proof, a uniform bound for the square of the ranks was obtained:

$$\sum_{f\in S_2(q)^*} r_f^2 \leq C_2 |S_2(q)^*|.$$

However, the latter improvement used only a slight variant of the methods of [KM1]. In fact, it is possible to pursue this idea further and it turns out that much more is true; this is the subject of the present note.

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