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LOCAL VANISHING OF CHARACTERISTIC COHOMOLOGY

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Introduction. To a smooth manifold M one can associate in a natural way a new smooth manifold, the manifold of k-jets of n-dimensional submanifolds of M, indicated by $G_n^{(k)}(M)$, which parametrizes in a smooth way the k-jets of immersed submanifolds of M. On $G_n^{(k)}(M)$ one can build in a canonical way a *differential ideal*, denoted $\mathcal{I}^{(k)}$. The cohomology associated to the complex $G_n^{(k)}(M)/\mathcal{I}^{(k)}$ is called *characteristic cohomology*. These ideas, which in part go back to [C], are explained here. A more detailed introduction to them can be read, for example, in the introduction to [BG1] or in [BGH1] (see also [BG2], [BGH2], [BGH3]). Characteristic cohomology appears in this picture as a cohomological tool to study n-dimensional submanifolds of M. In this context one should think of submanifolds as solutions to systems of PDEs (partial differential equations). For example, they could be integral manifolds of an integrable distribution or of a differential ideal. Characteristic cohomology (or a variation of it) can then be used to provide invariants for the system of PDEs. The notation for the qth characteristic cohomology group over a smooth manifold M is

$$H^q\left(\Omega^*\left(G_n^{(k)}(M)\right)/\mathcal{I}^{(k)}, d\right).$$

A first step in the study of characteristic cohomology is to see if something such as a Poincaré lemma holds for it; that is, if the characteristic cohomology vanishes on contractible open subsets of $G_n^{(k)}(M)$. More precisely, it has been conjectured by Griffiths [Gri] that for any contractible open set $\mathcal{U} \subset G_n^{(k)}(M)$ one has

$$H^q\left(\Omega^*(\mathcal{U})/\mathcal{I}^{(k)}(\mathcal{U}), d\right) = (0) \quad \text{when } 0 \le q < n.$$

For $\dim(M) - n = k = 1$ the result is classical. Griffiths and his collaborators [Gri] proved it when $n = 1$ and $k, \dim(M)$ is arbitrary, when $k = 1$ and $n, \dim(M)$ is arbitrary, and finally when $\dim(M) = 3$, $k = 2$, and $n = 2$. In Section 5, we prove precisely what was conjectured for all k, n, and $\dim(M)$.

For index $q = n$, the result is no longer true. This is because on a smooth manifold, one can have many independent functionals on n-dimensional submanifolds, even locally. The extremely rich structure of the nth characteristic cohomology group is probably the next object to research. Some intriguing conjectures on this topic, formulated by Griffiths, suggest possible approaches. Following this line of ideas,

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