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INTEGRALITY AND SYMMETRY OF QUANTUM LINK INVARIANTS

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0. Introduction. Quantum invariants of framed links whose components are colored by modules of a simple Lie algebra $\mathfrak{g}$ are Laurent polynomials in $v^{1/D}$ (with integer coefficients), where v is the quantum parameter and D an integer depending on $\mathfrak{g}$. We show that quantum invariants, with a suitable normalization, are Laurent polynomials in v^2.

We also establish two symmetry properties of quantum link invariants at roots of unity. The first asserts that quantum link invariants, at rth roots of unity, are invariant under the action of the affine Weyl group W_r, which acts on the weight lattice. A fundamental domain of W_r is the fundamental alcove $\bar{C}_r$, a simplex. Let G be the center of the corresponding simply connected complex Lie group. There is a natural action of G on $\bar{C}_r$. The second symmetry property, in its simplest form, asserts that quantum link invariants are invariant under the action of G if the link has zero linking matrix. The second symmetry property generalizes symmetry principles of Kirby and Melvin (the sl_2 case) and Kohno and Takata (the sl_n case) to arbitrary simple Lie algebra.

0.1. Quantum invariants. Suppose L is a framed link with m ordered components and $M_1, \dots, M_m$ are modules of a simple complex Lie algebra $\mathfrak{g}$. Then the quantum invariant $J_L(M_1, \dots, M_m)$ is a rational function in the variable $v^{1/D}$, where v is the *quantum parameter* and D is a number depending on $\mathfrak{g}$. (See [RT1], [Tu]; we recall the definition of quantum invariants in §1.) The Jones polynomial (see [Jo]) is the simplest in the family of quantum link invariants: When $\mathfrak{g} = \mathrm{sl}_2$ and the modules equal the fundamental representation, J_L is the Jones polynomial, with a suitable change of variable. The reader should be able to relate v to any other variable if it is known that the quantum integer $[n]$ is given by

$$[n] = \frac{v^n - v^{-n}}{v - v^{-1}}.$$

0.2. Integrality. A priori J_L is a rational function in $v^{1/D}$. Lusztig's result on the integrality of the R-matrix implies that J_L is a Laurent polynomial in $v^{1/D}$ with *integer coefficients* (see a detailed proof in §1.4.2 below). We study the integrality of the *exponents* of v. One of our main results shows that J_L is essentially a Laurent

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