Vol. 101, No. 3 DUKE MATHEMATICAL JOURNAL © 2000

A LEFSCHETZ (1,1) THEOREM FOR NORMAL PROJECTIVE COMPLEX VARIETIES

J. BISWAS AND V. SRINIVAS

1. Introduction. Let X be a projective variety over $\mathbb{C}$. Let X_{an} be the analytic space associated to X. Let $c_1 : \text{Pic}(X) \to H^2(X_{\text{an}}, \mathbb{Z})$ be the map that associates to a line bundle (or equivalently a Cartier divisor) on X its cohomology class. We may identify the Néron-Severi group $NS(X)$ with the image of $\text{Pic}(X)$ in $H^2(X_{\text{an}}, \mathbb{Z})$ under the above map.

If X is smooth, then by the Hodge decomposition theorem, we know that

$$H^2\big(X_{\text{an}}, \mathbb{C}\big) = H^{2,0}\big(X_{\text{an}}\big) \oplus H^{1,1}\big(X_{\text{an}}\big) \oplus H^{0,2}\big(X_{\text{an}}\big).$$

Let $F^1H^2(X_{\text{an}}, \mathbb{C}) = H^{2,0}(X_{\text{an}}) \oplus H^{1,1}(X_{\text{an}})$. The Lefschetz theorem on $(1,1)$ classes (see [GH], [L]) states that if X is a smooth, projective variety, then

$$NS(X) = \big\{\alpha \in H^2(X_{\text{an}}, \mathbb{Z}) \mid \alpha_{\mathbb{C}} \in F^1H^2\big(X_{\text{an}}, \mathbb{C}\big)\big\}.$$

If X is an arbitrary singular variety, then by [D, Theorem 8.2.2], the cohomology groups of X with $\mathbb{Z}$-coefficients carry mixed Hodge structures. Hence it makes sense to talk of $F^1H^2(X_{\text{an}}, \mathbb{C})$ for such a variety X. Spencer Bloch, in a letter to Jannsen [J, Appendix A], asks whether the "obvious" extension of the Lefschetz $(1,1)$ theorem is true for singular projective varieties, that is, is it true that

$$NS(X) = \big\{\alpha \in H^2\big(X_{\text{an}}, \mathbb{Z}\big) \mid \alpha_{\mathbb{C}} \in F^1H^2\big(X_{\text{an}}, \mathbb{C}\big)\big\}?$$

Barbieri Viale and Srinivas [BS2] give a counterexample to this question. Let X be a surface defined by the homogenous equation $w(x^3 - y^2z) + f(x, y, z) = 0$ in $\mathbb{P}^3_{\mathbb{C}}$, where x, y, z, w are homogenous coordinates in $\mathbb{P}^3_{\mathbb{C}}$ and f is a "general" homogenous polynomial over $\mathbb{C}$ of degree 4. They show that for such an X,

$$NS(X) \subsetneq \big\{\alpha \in H^2\big(X_{\text{an}}, \mathbb{Z}\big) \mid \alpha_{\mathbb{C}} \in F^1H^2\big(X_{\text{an}}, \mathbb{C}\big)\big\}.$$

In the same paper [BS2], the authors ask the following question. Let X be a complete variety over $\mathbb{C}$. Let $H^1(X, \mathcal{H}^1_X)$ be the subgroup of $H^2(X_{\text{an}}, \mathbb{Z})$ consisting of Zariski–locally trivial cohomology classes, that is, $\eta \in H^2(X_{\text{an}}, \mathbb{Z})$ lies in $H^1(X, \mathcal{H}^1_X)$ if and only if there exists a finite open cover $\{U_i\}$ of X by Zariski open sets such that $\eta \mapsto 0$ under the restriction maps $H^2(X_{\text{an}}, \mathbb{Z}) \to H^2((U_i)_{\text{an}}, \mathbb{Z})$ for all i. Is

$$NS(X) = \big\{\alpha \in H^1\big(X, \mathcal{H}^1_X\big) \mid \alpha_{\mathbb{C}} \in F^1H^2\big(X_{\text{an}}, \mathbb{C}\big)\big\}?$$

Received 13 April 1999.
1991 *Mathematics Subject Classification*. Primary 14C25, 14C30; Secondary 32J25, 32S35.