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CONFORMAL GEOMETRY, CONTACT GEOMETRY, AND THE CALCULUS OF VARIATIONS

JEFF A. VIACLOVSKY

1. Introduction. In the following we let (N, g_0) denote a compact, connected smooth Riemannian manifold of dimension $n \geq 3$. We denote the Ricci tensor and scalar curvature by *Ric* and R, respectively. In this paper we examine the nonlinear curvature equations

$$\sigma_k\left(Ric_g - \frac{R_g}{2(n-1)} \cdot g\right) = \text{constant} \tag{1}$$

for metrics g in the conformal class of g_0, where we use the metric g to view the tensor as an endomorphism of the tangent bundle and where σ_k denotes the trace of the induced map on the kth exterior power; that is, σ_k is the kth elementary symmetric function of the eigenvalues. The case $k = 1$, $R =$ constant is known as the Yamabe problem, and it has been studied in great depth (see [11] and [17]). We let $\mathcal{M}_1$ denote the set of unit volume metrics in the conformal class $[g_0]$. We show that these equations have the following variational properties.

THEOREM 1. *If $k \neq n/2$ and $(N, [g_0])$ is locally conformally flat, then a metric $g \in \mathcal{M}_1$ is a critical point of the functional*

$$\mathcal{F}_k : g \mapsto \int_N \sigma_k\left(Ric_g - \frac{R_g}{2(n-1)} \cdot g\right) dvol_g$$

restricted to $\mathcal{M}_1$ if and only if

$$\sigma_k\left(Ric_g - \frac{R_g}{2(n-1)} \cdot g\right) = C_k$$

for some constant C_k. If N is not locally conformally flat, then the statement is true for $k = 1$ and $k = 2$.

We compute the second variation of the above functionals and use this to examine the behavior of the functionals near a critical point. In particular, we show that they are elliptic when the eigenvalues are restricted to lie in a certain cone (see Section 6). Following [4], we call such a solution admissible. We prove Theorem 2 ($k \neq n/2$).

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