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EXTREMAL MANIFOLDS AND HAUSDORFF DIMENSION

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1. Introduction. The recent proof by D. Y. Kleinbock and G. A. Margulis [11] of Sprindžuk's conjecture for smooth nondegenerate manifolds M means that the set $\mathscr{L}_v(M)$ of v-approximable points (this and other terminology is explained below) on M is of zero induced Lebesgue measure. This raises the question of its Hausdorff dimension. Bounds and indeed the exact dimension for manifolds satisfying a variety of arithmetic, geometric, and analytic conditions are known (see [2], [3], [5], [7]). In this paper ubiquity is used to obtain a lower bound for the Hausdorff dimension of a set more general than $\mathscr{L}_v(M)$ for any extremal C^1 manifold M. Hitherto volume estimates that depend on curvature conditions were used to overcome a "small denominators" problem. It turns out, however, that extremality, when combined with Fatou's lemma, is all that is needed. We begin with some notation.

Let $|x| = \max\{|x_1|, \dots, |x_n|\}$ denote the supremum norm or height of the point $x = (x_1, \dots, x_n)$ in n-dimensional Euclidean space $\mathbb{R}^n$, and denote its Euclidean norm by $|x|_2 = (x_1^2 + \cdots + x_n^2)^{1/2}$. Throughout, $\mathbf{q} = (q_1, \dots, q_n)$ is a vector in $\mathbb{Z}^n$, and $\mathbf{q} \cdot x = q_1 x_1 + \cdots + q_n x_n$ denotes the usual inner product. For positive numbers a, b, we use the Vinogradov notation $a \ll b$ and $b \gg a$ if $a = O(b)$. If $a \ll b \ll a$, we write $a \asymp b$. A point $x \in \mathbb{R}^n$ that satisfies

$$\|\mathbf{q} \cdot x\| < |\mathbf{q}|^{-v} \tag{1}$$

for infinitely many $\mathbf{q} \in \mathbb{Z}^n$ is called *v-approximable* ($\|x\|$ is the distance of the real number x from $\mathbb{Z}$). Let M be an m-dimensional manifold in $\mathbb{R}^n$. The set of v-approximable points in the manifold M is denoted by $\mathscr{L}_v(M)$. The manifold M is called *extremal* if for any $v > n$, $\mathscr{L}_v(M)$ has Lebesgue measure 0. Equivalently, by Khintchine's transference principle, M is extremal if the set $\mathscr{S}_w(M)$ of points $x \in M$ that are simultaneously w-approximable (i.e., for which

$$\|qx\| < |q|^{-w}$$

for infinitely many $q \in \mathbb{Z}$) is null (i.e., of measure zero) when $w > 1/n$. Khintchine's theorem implies that the real line is extremal, and the terminology reflects the fact that

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