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AN L^p–L^q ESTIMATE FOR RADON TRANSFORMS ASSOCIATED TO POLYNOMIALS

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Let $S(x, y)$ be a polynomial of degree $n \geq 2$ with real coefficients. Thus

$$S(x, y) = \sum_{d=0}^{n} \sum_{j+k=d} a_{jk} x^j y^k = \sum_{d=0}^{n} \sum_{k=0}^{d} a_{d-k,k} x^{d-k} y^k, \tag{1}$$

where x, y, and a_{jk} are real numbers. We always assume that $a_{1,n-1} \neq 0$ or $a_{n-1,1} \neq 0$. The Radon transform of f associated to the polynomial $S(x, y)$ is defined by

$$Rf(t, x) = \int_{-\infty}^{\infty} f\big(t + S(x, y), y\big)\psi(t, x, y)\,dy, \tag{2}$$

where $\psi \in C_c^\infty(\mathbf{R}^3)$ is a cutoff function. (For the background information on the well-developed theory of Radon transforms and related oscillatory integral operators, we refer the reader to the papers [P], [PS3], [S], [Se2], and the references contained there.)

When $S(x, y)$ is a homogeneous polynomial of degree n, that is, $a_{d-k,k} = 0$ for $d < n$ in (1), the operator R was studied by Phong and Stein [PS2] as a model for degenerate Radon transforms. They proved among other things that, if $a_{1,n-1} \neq 0$ and $a_{n-1,1} \neq 0$, then R is bounded from $L^p(\mathbf{R}^2)$ to $L^q(\mathbf{R}^2)$, when $(1/p, 1/q)$ is in the set τ defined as follows. First let Δ be the closed convex hull (a trapezoid) of the points $O = (0, 0)$, $A = (2/(n+1), 1/(n+1))$, $A' = (n/(n+1), (n-1)/(n+1))$, and $O' = (1, 1)$ in the plane (see Figure 1). Then τ is defined to be Δ minus the half-open segments $(O, A]$ and $[A', O')$. Phong and Stein also proved that, for R to be bounded from $L^p(\mathbf{R}^2)$ to $L^q(\mathbf{R}^2)$, it is necessary that $(1/p, 1/q) \in \Delta$. When $n = 2$, 3, it is known that R is bounded precisely in Δ (see [PS1], [Se1], and [Se2]).

When $n \geq 4$, the endpoint (L^p, L^q) estimates have been open except in the translation-invariant case $S(x, y) = c(x - y)^n$, for which R is known to be bounded in all of Δ (see [PS2, p. 720] and [B]). (We want to point out here that the new method in [C2] may be used to prove a restricted weak type at the endpoints.)

The purpose of this paper is to give a positive answer to these remaining endpoint questions. In fact, our results are somewhat more general. Since homogeneity plays

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