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A MODULAR INVARIANCE ON THE THETA FUNCTIONS DEFINED ON VERTEX OPERATOR ALGEBRAS

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To Professor Toshiro Tsuzuku on his seventieth birthday

1. Introduction. Throughout this paper, V denotes a vertex operator algebra, or VOA, $(\oplus_{n=0}^{\infty} V_n, Y, \mathbf{1}, \omega)$ with central charge c and $Y(v,z)=\sum v(n)z^{-n-1}$ denotes a vertex operator of v. (Abusing the notation, we also use it for vertex operators of v for V-modules.) $o(v)$ denotes the grade-keeping operator of v, which is given by $v(m-1)$ for $v \in V_m$ and defined by extending it for all elements of V linearly. In particular, $o(\omega)$ equals $L(0)=\omega(1)$ for the Virasoro element ω of V and $o(v)=v(0)$ for $v \in V_1$. In order to simplify the situation, we assume that $\dim V_0 = 1$ so that there is a constant $\langle v,u\rangle \in \mathbb{C}$ such that $v_1 u = -\langle v,u\rangle \mathbf{1}$ for $v,u \in V_1$.

We call V a rational vertex operator algebra in the case when each V-module is a direct sum of simple modules. Define $C_2(V)$ to be the subspace of V spanned by elements $u(-2)v$ for $u,v \in V$. We say that V satisfies *condition* C_2 if $C_2(V)$ has finite codimension in V. For a V-module M with grading $M=\oplus M_m$, we define the formal character as

$$\mathrm{ch}_q M = q^{-c/24}\sum \dim M_m q^m = \mathrm{tr}_M\, q^{-c/24+L(0)}. \tag{1}$$

In this paper, we consider these functions less formally by taking q to be the usual local parameter $q=q_\tau=e^{2\pi i\tau}$ at infinity in the upper half-plane

$$\mathcal{H}=\left\{\tau\in\mathbb{C} \mid \mathcal{I}\tau>0\right\}.$$

Although it is often said that a VOA is a conformal field theory with mathematically rigorous axioms, the axioms of VOA do not assume the modular invariance. However, Zhu [Z] showed the modular ($\mathrm{SL}_2(\mathbb{Z})$) invariance of the space

$$\left\langle q_1^{|a_1|}\cdots q_n^{|a_n|}\,\mathrm{tr}_W\, Y(a_1,q_1)\cdots Y(a_n,q_n)q^{L(0)-c/24} : W \text{ irreducible } V\text{-modules}\right\rangle \tag{2}$$

for a rational VOA V with central charge c and $a_i \in V_{|a_i|}$ under condition C_2, which are satisfied by many known examples, where $q_j = q_{z_j} = e^{2\pi i z_j}$ and $|a_i|$ denotes the

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