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S-ARITHMETICITY OF DISCRETE SUBGROUPS CONTAINING LATTICES IN HOROSPHERICAL SUBGROUPS

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0. Introduction. Let $\mathbb{Q}_p$ be the field of p-adic numbers, and let $\mathbb{Q}_\infty = \mathbb{R}$. Let $\mathbf{G}_p$ be a connected semisimple $\mathbb{Q}_p$-algebraic group. The unipotent radical of a proper parabolic $\mathbb{Q}_p$-subgroup of $\mathbf{G}_p$ is called a *horospherical* subgroup. Two horospherical subgroups are called *opposite* if they are the unipotent radicals of two opposite parabolic subgroups. In [5] and [6], we studied discrete subgroups generated by lattices in two opposite horospherical subgroups in a simple real algebraic group with real rank at least 2. This work was inspired by the following conjecture posed by G. Margulis.

Conjecture 0.1. *Let $\mathbf{G}$ be a connected semisimple $\mathbb{R}$-algebraic group such that $\mathbb{R}$-rank $(\mathbf{G}) \geq 2$, and let $\mathbf{U}_1$, $\mathbf{U}_2$ be a pair of opposite horospherical $\mathbb{R}$-subgroups of $\mathbf{G}$. For each $i = 1, 2$, let F_i be a lattice in $\mathbf{U}_i(\mathbb{R})$ such that $H \cap F_i$ is finite for any proper normal $\mathbb{R}$-subgroup H of G. If the subgroup generated by F_1 and F_2 is discrete, then it is an arithmetic lattice in $\mathbf{G}(\mathbb{R})$.*

We settled the conjecture in many cases, including the case when $\mathbf{G}$ is an absolutely simple real split group with $\mathbf{G}(\mathbb{R})$ not locally isomorphic to $\mathrm{SL}_3(\mathbb{R})$ (see [5]).

In this paper, we study a problem analogous to the conjecture in a product of real and p-adic algebraic groups. The following is a special case of the main theorem, Theorem 4.3.

Theorem 0.2. *Let S be a finite set of valuations of $\mathbb{Q}$ including the archimedean valuation ∞. For each $p \in S$, let $\mathbf{G}_p$ be a connected semisimple algebraic $\mathbb{Q}_p$-group without any $\mathbb{Q}_p$-anisotropic factors, and let $\mathbf{U}_{1p}$, $\mathbf{U}_{2p}$ be a pair of opposite horospherical subgroups of $\mathbf{G}_p$. Set $G = \prod_{p\in S} \mathbf{G}_p(\mathbb{Q}_p)$, $U_1 = \prod_{p\in S} \mathbf{U}_{1p}(\mathbb{Q}_p)$, and $U_2 = \prod_{p\in S} \mathbf{U}_{2p}(\mathbb{Q}_p)$.*

Assume that $\mathbf{G}_\infty$ is absolutely simple $\mathbb{R}$-split with rank at least 2 and that if $\mathbf{G}_\infty(\mathbb{R})$ is locally isomorphic to $\mathrm{SL}_3(\mathbb{R})$, then $\mathbf{U}_{1\infty}$ is not the unipotent radical of a Borel subgroup of $\mathbf{G}_\infty$. Let F_1 and F_2 be lattices in U_1 and U_2, respectively. If the subgroup generated by F_1 and F_2 is discrete, then it is a nonuniform S-arithmetic lattice in G.

If p is a nonarchimedean valuation of $\mathbb{Q}$, then no horospherical subgroup of $\mathbf{G}_p(\mathbb{Q}_p)$ admits a lattice. Moreover, there is no infinite unipotent discrete subgroup in a p-adic Lie group. Therefore it is necessary to assume in Theorem 0.2 that S contains the archimedean valuation ∞.

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