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A NEW FORMULA FOR WEIGHT MULTIPLICITIES AND CHARACTERS

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1. Introduction. The weight multiplicities of a representation of a simple Lie algebra $\mathfrak{g}$ are the dimensions of eigenspaces with respect to a Cartan subalgebra $\mathfrak{h}$. In this paper, we give a new formula for these multiplicities.

Our formula expresses the multiplicities as sums of positive rational numbers. Thus it is very different from the classical formulas of Freudenthal [F] and Kostant [Ks], which express them as sums of positive and negative integers. It is also quite different from recent formulas due to Lusztig [L1] and Littelmann [Li].

For example, for the multiplicity of the next-to-highest weight in the n-dimensional representation of $\mathfrak{sl}_2$, we get the following expression (which sums to 1):

$$\frac{1}{(1)(2)}+\frac{1}{(2)(3)}+\cdots+\frac{1}{(n-1)(n)}+\frac{1}{n}.$$

The key role in our formula is played by the *dual* affine Weyl group.

Let V_0, $(\ ,\ )$ be the real Euclidean space spanned by the root system R_0 of $\mathfrak{g}$, and let V be the space of affine linear functions on V_0. We identify V with $\mathbb{R}\delta\oplus V_0$ via the pairing $(r\delta+x, y)=r+(x,y)$ for $r\in\mathbb{R}$, x, $y\in V_0$.

The dual affine root system is $R=\{m\delta+\alpha^\vee \mid m\in\mathbb{Z},\ \alpha\in R_0\}\subseteq V$, where $\alpha^\vee$ means $2\alpha/(\alpha,\alpha)$ as usual. Fix a positive subsystem $R_0^+\subseteq R_0$ with base $\{\alpha_1,\dots,\alpha_n\}$, and let β be the highest *short* root. Then a base for R is given by $a_0=\delta-\beta^\vee$, $a_1=\alpha_1^\vee,\dots,a_n=\alpha_n^\vee$, and we write s_i for the (affine) reflection about the hyperplane $\{x \mid (a_i,x)=0\}\subseteq V_0$.

The dual affine Weyl group is the Coxeter group W generated by $s_0,\dots,s_n$, and the finite Weyl group is the subgroup W_0 generated by $s_1,\dots,s_n$. For $w\in W$, its *length* is the length of a reduced (i.e., shortest) expression of w in terms of the s_i. The group W acts on the weight lattice P of $\mathfrak{g}$, and each orbit contains a unique (minuscule) weight from the set

$$\mathbb{O}:=\left\{\lambda\in P \mid \left(\alpha^\vee,\lambda\right)=0 \text{ or } 1,\ \forall\alpha\in R_0^+\right\}.$$

Definition. For each λ in P, we define

(1) $\widetilde{\lambda}:=\lambda+(1/2)\sum_{\alpha\in R_0^+}\varepsilon_{(\alpha^\vee,\lambda)}\alpha$, where, for $t\in\mathbb{R}$, ε_t is 1 if $t>0$ and -1 if $t\le 0$;

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