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APPROXIMATE SPECTRAL SYNTHESIS IN THE BERGMAN SPACE

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1. Introduction. From a hard-analysis point of view, the main result of the present paper is an approximation theorem for extremal (Bergman-inner) functions in terms of finite zero divisors. This corresponds to the famous Carathéodory-Schur theorem from the early 1900s, which states that every function in the unit ball of H^∞ (the uniform algebra of bounded holomorphic functions in the unit disk) can be approximated by finite Blaschke products, in the topology of uniform convergence on compact subsets of the disk. Along the way, we find a theorem about kernel functions for weighted Bergman spaces of the type that estimate them away from the diagonal.

The motivation for these results is the study of z-invariant subspaces in the Bergman space. It is known that the lattice of z-invariant subspaces in Bergman spaces has a very complicated structure. But one may single out among all z-invariant subspaces those of simplest nature, the zero-based ones. What z-invariant subspaces can be approximated by zero-based ones? What z^*-invariant subspaces can be approximated by finite-dimensional ones? In this paper, we answer these questions (see Theorems 1 and 2), and we discuss some relations with rational approximation and cyclic vectors for the backward shift.

Let X be a Banach space of functions analytic in the unit disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ of the complex plane $\mathbb{C}$. Suppose that X is invariant with respect to the operator M_z of multiplication by the independent variable. Many important problems concerning the structure of the lattice of subspaces of X that are closed and invariant with respect to M_z (or, simply, z-invariant) are related to problems of spectral synthesis. Let Y be the space dual to X. Then the functionals k_λ of evaluation of functions in X at the points of $\mathbb{D}$,

$$k_\lambda : f \to f(\lambda); \quad \lambda \in \mathbb{D},$$

are eigenvectors of the operator M_z^*:

$$\big(\lambda I - M_z^*\big) k_\lambda = 0;$$

the functionals $k_\lambda^{(n)}$ of evaluation of the derivatives

$$k_\lambda^{(n)} : f \to f^{(n)}(\lambda); \quad n \in \mathbb{Z}_+,\ \lambda \in \mathbb{D},$$

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